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DIFFRACTION OF A DIPOLE FIELD BY A UNIDIRECTIONALLY CONDUCTING SEMI-INFINITE SCREEN*

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Abstract. An exact solution is obtained for the diffraction of a dipole field by a unidirectionally conducting semi-infinite plane screen. Double Laplace transforms are applied to Maxwell's equations, and the defining conditions of the unidirectionality lead to an equation between two complex functions of two complex variables. This equation is solved by an extension of the usual function-theoretical method, and we can then express the electro-magnetic field in terms of certain complex triple integrals. These are transformed into real integrals, so that it is possible to discuss the field behavior in the neighborhood of the diffracting edge. The variation of singularity along the edge of the screen is given.

1. Introduction. Diffraction by a unidirectionally conducting body is the subject of two recent investigations: Toraldo di Francia [1] has given approximate results, based on a physical discussion, for a unidirectional screen of small diameter, while Karp [2] has obtained an exact solution for the diffraction of a plane wave by a unidirectional semi-infinite screen. Provided only the far field were of interest, well-known reciprocity considerations would suffice to extend the plane wave result to the case of dipole incidence; but a discussion of the near field, and in particular of the physically interesting variation of the fields and currents along the edge of the unidirectional screen, requires the complete solution of the diffraction problem for a dipole, and this is the problem here considered.

Important applications of the theory of unidirectional screens are: (1) to the measurement of the angular momentum of electromagnetic radiation, as in Toraldo di Francia's work (see [1, 9, 10]); (2) to microwave problems involving unidirectionally conducting components; (3) to problems of propagation over anisotropic media. In any of these cases, the problem of this paper plays the role of a canonical problem, in the sense that our results as to edge behavior for a semi-infinite screen permit (by way of Keller's geometrical theory of diffraction: see [11], and references given there to earlier work by Keller and his collaborators) the deduction of asymptotic results for a large class of unidirectional screens of finite size.

Our analysis is based on a formulation of the diffraction of a dipole field by a uni-

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directional semi-infinite screen as a Wiener-Hopf problem. Jones [3] observed that diffraction problems leading to Wiener-Hopf equations are advantageously treated by taking the transform of the differential equation before applying boundary conditions: extending Jones' method to the case of dipole incidence amounts to little more than replacing single by double transforms. The known solution [4] for a perfectly conducting screen may very easily be derived in this manner.

For the case of dipole incidence on a unidirectional semi-infinite screen, it is found after taking two successive Laplace transforms of the Maxwell equation

$$(\Delta_{xyz} + k^2)\mathbf{e} = 0,$$

that we may express the double transform of the electric field as an unknown vector function of the transform variables. Obtaining relations among the components of this vector function from the remaining Maxwell equations, and then applying the boundary condition and jump conditions derived from the unidirectionality, we show in Sec. 2 that the diffraction problem is equivalent to the solution of a single transform equation for two unknown complex functions. The transform equation also involves two independent complex variables and its solution therefore requires some modification of the usual function-theoretical considerations. This solution is derived in Sec. 3, with the result that the field components are expressed as Laplace inverses.

In Sec. 4, it is found that the originals of these inverse transforms are integrals of a type introduced by Macdonald [5], plus additional terms which it is possible to transform into certain real integrals. These results permit us to give the variation of the near field as the diffracting edge is traversed.

The results are summarized in Sec. 5, in the form of a theorem, and it is verified that all conditions of the problem are met.

2. Derivation of the transform equation. We consider diffraction by a unidirectionally conducting semi-infinite screen: $x \geq 0, -\infty < y < \infty, z = 0$, where x, y, z form a right-hand rectangular coordinate system. Let $\mathbf{e}_0$, $\mathbf{h}_0$ be the electric and magnetic field vectors of an incident dipole field, which we take as an electric dipole with axis perpendicular to the screen, while remarking that the method which follows is also applicable to an arbitrarily oriented dipole. Locating our dipole at (x_0, y_0, z_0) with $z_0 > 0$, we may describe the incident field by the Hertz vector $(0, 0, \Pi_z)$, where $\Pi_z = e^{-ikR}/kR$ and

$$R = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}.$$

The corresponding electromagnetic field components are

$$\begin{cases} \mathbf{e}_0 = \left(\dfrac{\partial^2\Pi_z}{\partial z\,\partial x}, \dfrac{\partial^2\Pi_z}{\partial z\,\partial y}, \dfrac{\partial^2\Pi_z}{\partial z^2} + k^2\Pi_z\right) \\ \mathbf{h}_0 = i\omega\epsilon\left(+\dfrac{\partial\Pi_z}{\partial y}, -\dfrac{\partial\Pi_z}{\partial x}, 0\right). \end{cases} \tag{1}$$

Time dependence $e^{i\omega t}$ is understood.

Now denote by $\mathbf{e}_0 + \mathbf{e}$, $\mathbf{h}_0 + \mathbf{h}$ the total electric and magnetic fields resulting from the incidence of $\mathbf{e}_0$, $\mathbf{h}_0$ upon the given screen. Then the scattered field vectors $\mathbf{e}, \mathbf{h}$ satisfy the time-harmonic Maxwell equations

$$(\Delta_{xyz} + k^2)\mathbf{e} = 0, \tag{2}$$

$$\nabla\cdot\mathbf{e} = 0, \tag{3}$$

$$-i\omega\mu\mathbf{h} = \nabla \times \mathbf{e}, \tag{4}$$

subject to a set of conditions (boundary condition, two jump conditions and an edge condition) of which the first three are intended as a phenomenological description of the unidirectional conductivity of the diffracting screen. These conditions are conveniently stated in terms of field components in the direction ξ of conductivity and the direction η normal to ξ:

$$e_\xi = -e_{0\xi}\,, \tag{5}$$

$$[h_\xi] = h_\xi(z+0) - h_\xi(z-0) = 0, \text{ across the screen,} \tag{6}$$

$$[e_\eta] = 0, \text{ across the screen,} \tag{7}$$

$$[h_\eta] = 0, \text{ at the edge of the screen.} \tag{8}$$

We also assume that $\mathbf{e}$, $\mathbf{h}$ are integrable at the edge of the screen: this, with (8), yields a unique solution.

Conditions at infinity complete our specification of the scattered field. We impose the familiar condition that $\mathbf{e}$, $\mathbf{h}$ be exponentially damped solutions of the three-dimensional wave equation. The implied behavior at infinity is that of the elementary solution e^{-ikR}/kR, where we suppose k to have a negative imaginary part: $k = k_1 - ik_2$ $(k_2 > 0)$.

Now define the double Laplace transforms $\mathbf{E}$, $\mathbf{H}$ of the scattered field vectors $\mathbf{e}$, $\mathbf{h}$:

$$\begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{Bmatrix}\mathbf{e}\\ \mathbf{h}\end{Bmatrix} e^{-sx+ty}\,dx\,dy.$$

It follows from (2) that $\mathbf{E}$ satisfies

$$\left(\frac{\partial^2}{\partial z^2} + K^2\right)\mathbf{E} = 0, \tag{9}$$

where $K^2 = k^2 + s^2 + t^2$. Our condition on $\mathbf{e}$ at infinity yields the range of validity of (9): we fix K by the choice

$$(k^2 + s^2 + t^2)^{1/2}|_{s=t=0} = +k,$$

and note that (9) is meaningful for $|\operatorname{Re} t| < k_2$, $|\operatorname{Re} s| < |\operatorname{Im}(k^2+t^2)^{\frac{1}{2}}|$. In this domain of s, t we write the solution of (9) (notice $\operatorname{Im} K < 0$) as

$$\begin{aligned}\mathbf{E} &= \mathbf{A}(s,t)\exp[-iK(z+z_0)], \qquad z > 0\\ &= \mathbf{B}(s,t)\exp[iK(z-z_0)], \qquad z < 0.\end{aligned} \tag{10}$$

We proceed to determine the transform of the scattered electric field by solving for $\mathbf{A}(s, t)$, $\mathbf{B}(s, t)$.

It is convenient to rotate coordinates from x, y to ξ, η. Let α_0 $(0 < \alpha_0 < \pi/2)$ be the angle, measured clockwise, from the x-direction to the ξ-direction. Then

$$\begin{aligned}\xi &= x\cos\alpha_0 - y\sin\alpha_0\,,\\ \eta &= x\sin\alpha_0 + y\cos\alpha_0\,.\end{aligned} \tag{11}$$

The corresponding rotated transform variables p, q, with the property $p\xi + q\eta = sx - ty$, are given by

$$\begin{aligned} p &= s \cos \alpha_0 + t \sin \alpha_0 , \\ q &= s \sin \alpha_0 - t \cos \alpha_0 . \end{aligned} \tag{12}$$

We notice that $s^2 + t^2 = p^2 + q^2$ is invariant under the rotation, so that $K^2 = k^2 + s^2 + t^2 = k^2 + p^2 + q^2$. It is clear that (10) is unchanged if $\mathbf{A}$, $\mathbf{B}$ are understood to be functions of p, q rather than s, t.

We now deduce the basic transform equation of our problem. Observe first that (10) permits us to write the transform of (4) as

$$-i\omega\mu\mathbf{H} = (p\mathbf{i}_1 + q\mathbf{i}_2 \mp iK\mathbf{i}_3)\cdot\mathbf{E}, \tag{13}$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, are unit vectors in the ξ, η and z directions. Similarly, we write the transform of (3) as

$$(p\mathbf{i}_1 + q\mathbf{i}_2 \mp iK\mathbf{i}_3)\cdot\mathbf{E} = 0, \tag{14}$$

which according to (10) is equivalent to

$$\begin{aligned} pA_\xi + qA_\eta - iKA_z &= 0, \\ pB_\xi + qB_\eta + iKB_z &= 0. \end{aligned} \tag{15}$$

Since conditions (5), (7) yield $A_\xi = B_\xi$, $A_\eta = B_\eta$, we see that $B_z = -A_z$, while

$$A_z = \frac{1}{iK}(pA_\xi + qA_\eta). \tag{16}$$

The transform of (6), with $[H_\xi]$ computed from (13), then gives

$$qA_z = -iKA_\eta , \tag{17}$$

from which

$$A_\eta = \frac{pqA_\xi}{k^2 + p^2}. \tag{18}$$

Now calculate $[H_\eta]$ from (13). If we use (17), (18) to eliminate A_z, and if we denote $(i\omega\mu/2)\,[H_\eta] = \Lambda$, the result is our basic transform equation:

$$(k^2 + p^2)\Lambda = k^2 iKA_\xi \exp(-iKz_0). \tag{19}$$

Now simplify (19) by introducing the boundary condition (5). Let

$$\mathbf{E}_+ = \int_0^\infty dx \int_{-\infty}^\infty dy\, e^{-sx+ty}\mathbf{e}(x, y, z),$$

$$\mathbf{E}_- = \int_{-\infty}^0 dx \int_{-\infty}^\infty dy\, e^{-sx+ty}\mathbf{e}(x, y, z),$$

$$\mathbf{E}_\pm(s, t, z)\big|_{z=0} = \mathbf{E}_\pm(0)$$

and define the ξ-component of $\mathbf{E}_\pm(0)$ in accordance with (11) as

$$E_{\pm\xi}(0) = E_{\pm x}(0)\cos\alpha_0 - E_{\pm y}(0)\sin\alpha_0 .$$

Then

$$E_{+\xi}(0) + E_{-\xi}(0) = A_\xi \exp(-iKz_0), \tag{20}$$

where we see from (5) that

$$E_{+\xi}(0) = -\int_0^\infty dx \int_{-\infty}^\infty dy\, e^{-sx+ty} e_{0\xi}$$

and from (1) that

$$e_{0\xi} = \left(\frac{\partial^2 \Pi_z}{\partial z\, \partial x} \cos \alpha_0 - \frac{\partial^2 \Pi_z}{\partial z\, \partial y} \sin \alpha_0 \right) \bigg|_{z=0}. \tag{21}$$

Since Π_z differs only by a factor of $k/4\pi$ from the free-space Green's function of the three-dimensional wave equation, we readily deduce the y-transform (valid for $|\operatorname{Re} t| < k_2$)

$$\int_{-\infty}^\infty \frac{\exp[-ik\{(x - x_0)^2 + (y - y_0)^2 + z_0^2\}^{1/2} + ty]}{k\{(x - x_0)^2 + (y - y_0)^2 + z_0^2\}^{1/2}} dy = -\frac{\pi i}{k} e^{ty_0} H_0^{(2)}[K_0\{(x - x_0)^2 + z_0^2\}^{1/2}], \tag{22}$$

where

$$K_0 = (k^2 + t^2)^{1/2},$$

and

$$(k^2 + t^2)^{1/2}|_{t=0} = +k.$$

The x-transform

$$\int_0^\infty dx\, e^{-sx}$$

of the right side of (22) is then obtained from the known integral representation

$$H_0^{(2)}[K_0\{(x - x_0)^2 + z_0^2\}^{1/2}] = -\frac{1}{\pi i} \int_{w_0 - i\infty}^{w_0 + i\infty} \frac{\exp[w(x - x_0) - i(K_0^2 + w^2)^{1/2} z_0]}{(K_0^2 + w^2)^{1/2}} dw, \tag{23}$$

where $|w_0| < |\operatorname{Im} K_0|$. Recalling that $z_0 > 0$, we conclude that the double transform of (21), valid for $\operatorname{Re} s > w_0$, is

$$E_{+\xi}(0) = \frac{ipe^{ty_0}}{k} \int_{w_0 - i\infty}^{w_0 + i\infty} \frac{\exp[-wx_0 - i(K_0^2 + w^2)^{1/2} z_0]}{s - w} dw. \tag{24}$$

Applying (20) and (24), we put (19) in the form

$$(k^2 + p^2)\Lambda = -kpKe^{ty_0} \int_{w_0 - i\infty}^{w_0 + i\infty} \frac{\exp[-wx_0 - i(K_0^2 + w^2)^{1/2} z_0]}{s - w} dw + k^2 iKE_{-\xi}(0), \tag{25}$$

where $|w_0| < |\operatorname{Im} K_0|$. Either of the two unknown functions Λ, $E_{-\xi}(0)$ completely determines $\mathbf{E}$.

3. Transforms of the field components. The usual Wiener-Hopf techniques are not immediately applicable to (25), since the domains of regularity of the functions depend on the two complex variables s and t simultaneously. In this section, we show how the difficulty may be overcome by restricting all operations to a suitable range of t. For such t, (25) is treated as if s alone were the variable. Certain representation theorems

for the Laplace transform are applied to carry out the function-theoretical argument, and the resulting knowledge of Λ yields the transforms of the various field components.

Now all field components are required to be exponentially damped solutions of the three-dimensional wave equation. It follows that $\Lambda = (i\omega\mu/2)\ [H_\eta]$ is regular for Re $(s + iK_0) > 0$, while $E_{-\xi}(0)$ is regular for Re $(iK_0 - s) > 0$. The respective half-planes of regularity depend on $K_0 = (k^2 + t^2)^{1/2}$, but a discussion based on (22) shows that the same t-condition suffices for the regularity of both Λ and $E_{-\xi}(0)$, namely that t be in the strip $|\text{Re } t| < k_2$.

The factorization $K = (s + iK_0)^{1/2} (s - iK_0)^{1/2}$ then permits us to rewrite (25) as

$$kpe^{ty_0}(p - ik)^{-1}(s - iK_0)^{1/2} \int_{w_0 - i\infty}^{w_0 + i\infty} \exp\left[-wx_0 - i(K_0^2 + w^2)^{1/2}z_0\right](s - w)^{-1}\, dw$$
$$= -(p + ik)(s + iK_0)^{-1/2}\Lambda + ik^2(p - ik)^{-1}(s - iK_0)^{1/2}E_{-\xi}(0), \tag{26}$$

where $|w_0| < |\text{Im } K_0|$, and where the condition $|\text{Re } t| < k_2$ is met by choosing Re $t = 0$. Then $|\text{Im } K_0| \geq k_2$, and we may take $|w_0| < k_2$. Denote the left side of (26) by $f(s)$ and the first and second terms on the right by $g(s)$, $h(s)$ respectively. The equation

$$f(s) = g(s) + h(s), \tag{26'}$$

where $f(s)$ is regular in the strip $w_0 < \text{Re } s < k_2$, and $g(s)$, $h(s)$ are regular in the overlapping half-plane Re $s > -k_2$, Re $s < k_2$, may be solved for the unknown functions $g(s)$, $h(s)$ by an application of the Wiener-Hopf technique. The procedure appears to be equivalent to that given formally by Harrington [6], but the justification may perhaps be of interest.

We reason as follows. The fact that $f(s)$ is regular in a strip suggests that it is there represented by a two-sided Laplace transform. We shall assume such a representation: the assumption will be justified on the basis of results we obtain for $g(s)$ and $h(s)$. Provisionally, then, we write

$$f(s) = \int_{-\infty}^{\infty} e^{-sx}F(x)\, dx, \tag{27}$$

where

$$F(x) = \frac{1}{2\pi i}\int_{\zeta_0 - i\infty}^{\zeta_0 + i\infty} e^{x\zeta}f(\zeta)\, d\zeta \qquad (|\zeta_0| < k_2).$$

Next observe that $h(s)$ has the form

$$h(s) = ik^2 \sec \alpha_0 s^{-1/2}E_{-\xi}(0) + s^{-3/2}h_1(s),$$

where $h_1(s)$ is bounded for Re $s < k_2$. The function

$$s^{-3/2}h_1(s) = h(s) - ik^2 \sec \alpha_0 s^{-1/2}E_{-\xi}(0) \tag{28}$$

then meets all conditions of a standard representation theorem [7] for the one-sided Laplace transform. Since $ik^2 \sec \alpha_0\ s^{-1/2}\ E_{-\xi}(0)$ is a Laplace transform [namely the transform of the convolution of the inverses of $s^{-1/2}$ and of $ik^2 \sec \alpha_0\ E_{-\xi}(0)$], it follows that $h(s)$ is.

As to $g(s)$, we notice that Λ and $s^{-1/2}\ \Lambda$ are transforms. To consider $s\ \Lambda$, write

$$s\Lambda = \frac{i\omega\mu}{2}\lim_{\epsilon \to 0}\int_{\epsilon}^{\infty} \frac{\partial[h_\eta]}{\partial x}\, e^{-sx}\, dx. \tag{29}$$

An integration by parts, with an application of our edge condition (8) as $\epsilon \to 0$, permits us to conclude that $s\Lambda$ is a Laplace transform; an argument similar to that given for $h(s)$ then shows that $g(s)$ meets the conditions of the representation theorem [7] used above.

Our result (from which the assumed representation (27) follows directly) is that $g(s)$, $h(s)$ are Laplace transforms of certain functions $G(x)$, $H(x)$ over the ranges $(0, \infty)$, $(-\infty, 0)$ respectively. Adding these representations to the now proved (27), we replace (26′) by

$$\int_{-\infty}^{\infty} e^{-sx}F(x)\,dx = \int_0^{\infty} e^{-sx}G(x)\,dx + \int_{-\infty}^{0} e^{-sx}H(x)\,dx. \tag{30}$$

The standard uniqueness theorem [8] for two-sided transforms readily yields

$$\int_0^{\infty} e^{-sx}F(x)\,dx = g(s), \tag{31a}$$

$$\int_{-\infty}^{0} e^{-sx}F(x)\,dx = h(s). \tag{31b}$$

Our transform equation (26′) is therefore solved for the two unknown functions $g(s)$, $h(s)$. We remark that the familiar function-theoretic ingredients (factorization via Cauchy's integral formula, the appeal to analytic continuation and to Liouville's theorem) of the Wiener-Hopf techniques are implicit in the representation and uniqueness theorems we have used. At the same time, it appears that the implications and domain of applicability of the technique may be considerably enlarged by drawing on more general representation theorems, which are independent of the classical function-theoretic approach of Wiener and Hopf. We develop this point of view, and give applications to diffraction theory, in a forthcoming investigation.

Expressions for the transforms of the various field components follow at once from either (31a) or (31b). Let us apply (31a): the substitution of our definitions [see (26), (26′)] of $g(s)$, $f(s)$ and

$$F(x) = \frac{1}{2\pi i}\int_{\zeta_0 - i\infty}^{\zeta_0 + i\infty} f(\zeta)e^{\zeta x}\,d\zeta \qquad (|\zeta_0| < k_2)$$

in (31a) leads to

$$\Lambda = ke^{ty_0}(s + iK_0)^{1/2}(p + ik)^{-1}\cdot J, \tag{32}$$

where

$$J = \int_{w_0 - i\infty}^{w_0 + i\infty} \frac{(w\cos\alpha_0 + t\sin\alpha_0)(w - iK_0)^{1/2}}{(w - s)(w\cos\alpha_0 + t\sin\alpha_0 - ik)} \exp\left[-wx_0 - iz_0(K_0^2 + w^2)^{1/2}\right] dw \tag{33}$$

and $|w_0| < |\operatorname{Im} K_0|$. We then apply (19), (18), (17) in succession to find

$$A_\xi \exp(-ikz_0) = \frac{\exp(ty_0)(p - ik)}{ik(s - iK_0)^{1/2}}\cdot J, \tag{34}$$

$$A_\eta \exp(-ikz_0) = \frac{pq\exp(ty_0)}{ik(p + ik)(s - iK_0)^{1/2}}\cdot J, \tag{35}$$

$$A_z \exp(-ikz_0) = \frac{-p\exp(ty_0)(s + iK_0)^{1/2}}{k(p + ik)}\cdot J. \tag{36}$$

The $x-$ and $y-$ components of $\mathbf{A}(x, t)$ follow from (34), (35):

$$A_x \exp(-iKz_0) = \frac{\exp(ty_0)\{(k^2 + p^2)\cos\alpha_0 + pq\sin\alpha_0\}}{ik(p + ik)(s - iK_0)^{1/2}} \cdot J, \tag{37}$$

$$A_y \exp(-iKz_0) = \frac{-\exp(ty_0)\{(k^2 + p^2)\sin\alpha_0 - pq\cos\alpha_0\}}{ik(p + ik)(s - iK_0)^{1/2}} \cdot J. \tag{38}$$

In accordance with (10), the scattered electric field is given by

$$\mathbf{e} = \frac{1}{(2\pi i)^2}\int_S\int_T \{\mathbf{A}(s, t)\exp(-iKz_0)\}\exp[sx - i\mid z\mid K - ty]\,dt\,ds, \tag{39}$$

where S denotes a vertical contour from $s_0 - i\infty$ to $s_0 + i\infty$, with $|s_0| < k_2$, while T denotes a vertical contour from $-i\infty$ to $+i\infty$. We now substitute (34), (35), (36) in (39) to obtain explicit expressions for the field components e_ξ, e_η, e_z. Writing W for the vertical contour from $w_0 - i\infty$ to $w_0 + i\infty$, where $|w_0| < |\operatorname{Im} K_0|$, and denoting $(w\cos\alpha_0 + t\sin\alpha_0)$ by $-p_0$, we introduce the complex integrals

$$I = -\frac{1}{(2\pi i)^2 k}\int_S\int_T\int_W \frac{pp_0}{(p + ik)(p_0 + ik)} \cdot \frac{\exp[sx - wx_0 - iK\mid z\mid - i(K_0^2 + w^2)^{1/2}z_0 - t(y - y_0)]}{(s - w)\{(s - iK_0)(w + iK_0)\}^{1/2}}\,dw\,dt\,ds \tag{40}$$

and

$$G = \frac{-k}{(2\pi i)^2}\int_S\int_T\int_W \frac{\exp[sx - wx_0 - iK\mid z\mid - i(K_0^2 + w^2)^{1/2}z_0 - t(y - y_0)]}{(s - w)(p + ik)(p_0 + ik)\{(s - iK_0)(w + iK_0)\}^{1/2}}\,dw\,dt\,ds \tag{41}$$

in terms of which we express the electric field components as

$$e_\xi = I_{\xi z_0} + G_{\xi_0 z_0}, \tag{42}$$

$$e_\eta = \frac{1}{k^2}G_{\xi\xi_0\eta z_0}, \tag{43}$$

$$e_z = \frac{1}{k^2}G_{\xi\xi_0 z z_0}, \tag{44}$$

where the subscripts of I, G denote partial derivatives, with the ξ and ξ_0 derivatives defined by

$$\frac{\partial}{\partial\xi} = \frac{\partial}{\partial x}\cos\alpha_0 - \frac{\partial}{\partial y}\sin\alpha_0, \tag{45}$$

$$\frac{\partial}{\partial\xi_0} = \frac{\partial}{\partial x_0}\cos\alpha_0 - \frac{\partial}{\partial y_0}\sin\alpha_0. \tag{46}$$

The operational equivalents of $\partial/\partial\xi$, $\partial/\partial\xi_0$ are accordingly p, p_0 respectively.

Applying (4), the components of the scattered magnetic field are

$$h_\xi = 0, \tag{47}$$

$$h_\eta = -\frac{1}{i\omega\mu}G_{\xi_0 z z_0} \tag{48}$$

$$h_z = \frac{1}{i\omega\mu}G_{\xi_0\eta z_0}. \tag{49}$$

The x- and y-components of the field are linear combinations of the ξ- and η-components:

$$e_x = e_\xi \cos\alpha_0 + e_\eta \sin\alpha_0$$

$$e_y = -e_\xi \sin\alpha_0 + e_\eta \cos\alpha_0$$

and

$$h_x = h_\xi \cos\alpha_0 + h_\eta \sin\alpha_0$$

$$h_y = -h_\xi \sin\alpha_0 + h_\eta \cos\alpha_0 .$$

4. Discussion of the field; edge behavior. In this section, we relate the integrals I, G of (40), (41) to an integral

$$F = \frac{-1}{(2\pi i)^2 k} \int_S \int_T \int_W \frac{\exp\,[sx - wx_0 - iK \mid z \mid - i(K_0^2 + w^2)^{1/2} z_0 - t(y - y_0)]}{(s - w)\{(s - iK_0)(w + iK_0)\}^{1/2}}\, dw\, dt\, ds \tag{50}$$

which may be calculated explicitly. It is in fact readily shown that $\Phi = e^{-ikR}/kR + F$ is exactly the classical solution of Macdonald [5] for the field of a point source in the presence of a semi-infinite screen on which the field vanishes. Macdonald's result

$$\Phi = I_R - I_S , \tag{51}$$

where

$$I_R = \frac{i}{2} \int_{-\mu_R}^{\infty} H_1^{(2)}(kR \cosh \mu)\, d\mu, \tag{52}$$

$$I_S = \frac{i}{2} \int_{-\mu_S}^{\infty} H_1^{(2)}(kS \cosh \mu)\, d\mu, \tag{53}$$

with

$$R = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2},$$

$$S = [(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2]^{1/2},$$

$$\mu_R = \sinh^{-1} \{(2/R)(rr_0)^{1/2} \cos [(\phi - \phi_0)/2]\},$$

$$\mu_S = \sinh^{-1} \{(2/S)(rr_0)^{1/2} \cos [(\phi + \phi_0)/2]\},$$

and

$$x = r\cos\phi, \qquad x_0 = r_0 \cos\phi_0 , \qquad z = r\sin\phi, \qquad z_0 = r_0 \sin\phi_0 ,$$

will therefore facilitate our discussion of the present solution.

Now consider the definitions (40), (41), (50) of I, G, F. We find

$$\frac{\partial^2 G}{\partial\xi\, \partial\xi_0} = k^2 I \tag{54}$$

and

$$\left(\frac{\partial}{\partial\xi} + ik\right)\left(\frac{\partial}{\partial\xi_0} + ik\right) G = k^2 F, \tag{55}$$

so that

$$I = F + G + \frac{1}{ik}\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\xi_0}\right) G. \tag{56}$$

Applying (54) to $\mathbf{e}$ as given by (42), (43), (44) we have

$$\mathbf{e} = (I_{\xi z_0} + G_{\xi_0 z_0}, I_{\eta z_0}, I_{z z_0}). \tag{57}$$

The result $F = -(e^{-ikR}/kR) + \Phi$, where Φ is Macdonald's fundamental solution as given by (51), (52), (53), is then combined with (56) to give

$$\mathbf{e} + \mathbf{e}_0 = \left\{\Phi_{\xi z_0} + \Psi^{(1)}, \Phi_{\eta z_0} + \Psi^{(2)}, \Phi_{z z_0} + k^2\left(\frac{e^{-ikR}}{kR}\right) + \Psi^{(3)}\right\}, \tag{58}$$

where

$$\Psi^{(1)} = G_{z_0 \xi} + G_{z_0 \xi_0} + \frac{1}{ik}(G_{z_0 \xi \xi_0} + G_{z_0 \xi \xi}), \tag{59i}$$

$$\Psi^{(2)} = G_{\eta z_0} + \frac{1}{ik}(G_{\eta z_0 \xi_0} + G_{\eta z_0 \xi}), \tag{59ii}$$

$$\Psi^{(3)} = G_{z z_0} + \frac{1}{ik}(G_{z z_0 \xi_0} + G_{z z_0 \xi}). \tag{59iii}$$

An explicit integral representation for $\Psi^{(1)}$ now follows from (41), (59i). We see first that

$$\left(\frac{\partial}{\partial \xi_0} + ik\right)\Psi^{(1)} = -i \cos \alpha_0 \frac{\partial M}{\partial z_0}$$

where

$$M = \frac{1}{(2\pi i)^2} \int_S \int_T \int_W \exp [sx - wx_0 - iK \mid z \mid$$

$$- i(K_0^2 + w^2)^{1/2} z_0 - t(y - y_0)] \frac{dw\, dt\, ds}{\{(s - iK_0)(w + iK_0)\}^{1/2}}.$$

But we may integrate over W, T, S successively to evaluate M. Consider

$$M_1 = \frac{1}{2\pi i} \int_W (w + iK_0)^{-1/2} \exp [-wx_0 - iz_0(K_0^2 + w^2)^{1/2}]\, dw.$$

Let $x_0 = r_0 \cos \phi_0$, $z_0 = r_0 \sin \phi_0$ and apply Cauchy's theorem to deform the path into $w = iK_0 \cos (\phi_0 + i\eta)$, with $-\infty < \eta < \infty$; the result is

$$M_1 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\exp (-iK_0 r_0 \cosh \eta) K_0 \sin (\phi + i\eta)\, d\eta}{(2iK_0)^{1/2} \cos [(\phi_0 + i\eta)/2]}$$

$$= \frac{e^{-i\pi/4}(2K_0)^{1/2}}{2\pi i} \int_{-\infty}^{\infty} \exp (-iK_0 r_0 \cosh \eta) \sin [(\phi + i\eta)/2]\, d\eta$$

$$= -i \sin (\phi_0/2) r_0^{-1/2} \exp (-iK_0 r_0).$$

Similarly, we deform S into $s = -ik \cos (\theta + i\eta)$ to obtain

$$M_2 = \frac{1}{2\pi i} \int_S (s - iK_0)^{-1/2} \exp [sx - iK \mid z \mid]\, ds = i \sin (\phi/2) r^{-1/2} \exp (-iK_0 r),$$

where $x = r\cos\phi$, $z = r\sin\phi$. We then have

$$M = \int_T (M_1)(M_2)\exp[-t(y-y_0)]\,dt$$
$$= (rr_0)^{-1/2}\sin(\phi/2)\sin(\phi_0/2)\int_T \exp[-t(y-y_0) - iK_0(r+r_0)]\,dt,$$

and, applying (23),

$$M = -\pi k(r+r_0)(rr_0)^{-1/2}\sin(\phi/2)\sin(\phi_0/2)\rho^{-1}H_1^{(2)}(k\rho),$$

with $\rho = [(y-y_0)^2 + (r+r_0)^2]^{1/2}$. Solving the differential equation

$$\left(\frac{\partial}{\partial\xi_0} + ik\right)\Psi^{(1)} = -i\cos\alpha_0\,\frac{\partial M}{\partial z_0},$$

we find

$$\Psi^{(1)} = r^{-1/2}\sin(\phi/2)\Big\{\pi ik\cos\alpha_0\sin(\phi_0/2)\exp(-ik\xi_0)$$
$$\cdot\frac{\partial}{\partial z_0}\int_{-\infty}^{\xi_0}\frac{(r+r_0')}{(r_0')^{1/2}}\frac{H_1^{(2)}(k\rho')}{\rho'}\exp(ik\xi_0')\,d\xi_0'\Big\}, \tag{60i}$$

where

$$x = r\cos\phi, \qquad z = r\sin\phi$$
$$r_0' = [(x_0')^2 + z_0^2]^{1/2}, \qquad x_0' = \xi_0'\cos\alpha_0 + \eta_0\sin\alpha_0$$
$$\rho' = [(y-y_0')^2 + (r+r_0')^2]^{1/2}, \qquad y_0' = -\xi_0'\sin\alpha_0 + \eta_0\cos\alpha_0\,.$$

It is to be noticed that $\Psi^{(1)}$ vanishes on the screen, as does Φ_{ξ_0}, so that our boundary condition (5) is satisfied.

Operational considerations now permit us to evaluate $\Psi^{(2)}$, $\Psi^{(3)}$. Let G_* denote the image of G under our two-dimensional transform; the operational equivalences

$$\Psi^{(1)} \doteq (-1/k)(K_0^2 + w^2)^{1/2}(p + p_0 + ik)G_*\,,$$
$$\Psi^{(2)} \doteq (-1/k)(K_0^2 + w^2)^{1/2}q(p + p_0 + ik)G_*\,,$$
$$\Psi^{(3)} \doteq (i/k)\{(K_0^2 + w^2)(K_0^2 + s^2)\}^{1/2}(p + p_0 + ik)G_*\,,$$

follow from (59i), (59ii), (59iii). Then let $\Psi_*^{(1)}$ denote the image of $\Psi^{(1)}$; the equivalences

$$\Psi^{(2)} \doteq \frac{q}{p+ik}\Psi_*^{(1)}$$
$$\Psi^{(3)} \doteq \frac{-i(K_0^2 + s^2)^{1/2}}{p+ik}\Psi_*^{(1)}$$

lead to evaluations

$$\Psi^{(2)} = e^{-ik\xi}\int_{-\infty}^{\xi}\exp(ik\xi')\left.\frac{\partial\Psi^{(1)}}{\partial\eta}\right|_{\xi=\xi'}d\xi' \tag{60ii}$$

$$\Psi^{(3)} = e^{-ik\xi}\int_{-\infty}^{\xi}\exp(ik\xi')\left.\frac{\partial\Psi^{(1)}}{\partial z}\right|_{\xi=\xi'}d\xi', \tag{60iii}$$

where $\Psi^{(1)}$ is given by (60i). It is clear from (60ii) that $\Psi^{(2)}$ is an even function of z and therefore continuous across the screen: since $\Phi_{\eta z_0}$ vanishes on the screen, it follows that the jump condition (7) is satisfied.

The behavior of $\Psi^{(1)}$ in the neighborhood of the edge of the screen follows from (60i). As $r \to 0$, the behavior is

$$\Psi^{(1)} \sim r^{-1/2} \sin (\phi/2) v(y), \tag{61i}$$

where

$$v(y) = \pi ik \cos \alpha_0 \exp(-ik\xi_0) \frac{\partial}{\partial z_0} \int_{-\infty}^{\xi_0} (r_0')^{1/2} \sin (\phi_0'/2) \left. \frac{H_1^{(2)}(k\rho')}{\rho'} \right|_{r=0} \exp(ik\xi_0')\, d\xi_0' .$$

The leading terms in the expansions of $\Psi^{(2)}$, $\Psi^{(3)}$ are:

$$\Psi^{(2)} \sim r^{-1/2} \sin (\phi/2) v(y) \tan \alpha_0 \tag{61ii}$$

$$\Psi^{(3)} \sim -r^{-1/2} \cos (\phi/2) v(y) \sec \alpha_0 . \tag{61iii}$$

For the net edge behavior of the electric field components, we see from (58) that we must add terms arising from the differentiation of Φ to the $\Psi^{(1)}$, $\Psi^{(2)}$, $\Psi^{(3)}$ terms. The terms to be added are obtained from the expansion

$$\Phi \simeq 2\pi (rr_0)^{1/2} \sin (\phi/2)(\sin \phi_0/2)(\rho_0)^{-1} H_1^{(2)}(k\rho_0)$$

($\rho_0 = \rho \,|_{r=0}$), which follows from our expression for F [see (50)] upon our noticing that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_0}\right) F = -\frac{M}{k} = \frac{\pi (r + r_0) \sin (\phi/2) \sin (\phi_0/2)}{(rr_0)^{1/2}} \frac{H_1^{(2)}(k\rho)}{\rho}.$$

The explicit results for the leading terms are

$$e_\xi \sim r^{-1/2} \sin (\phi/2) \left(\frac{i}{k} \frac{\partial v}{\partial \xi_0} \right), \tag{62}$$

$$e_\eta \sim r^{-1/2} \sin (\phi/2) \left(\frac{i \tan \alpha_0}{k} \frac{\partial v}{\partial \xi_0} \right), \tag{63}$$

$$e_z \sim r^{-1/2} \cos (\phi/2) \left(\frac{\sec \alpha_0}{ik} \frac{\partial v}{\partial \xi_0} \right), \tag{64}$$

where $v(y)$ is given above under (61i).

To complete our description of the field near the edge of the screen, we consider (47), (48), (49). The results for the magnetic field components as $r \to 0$ are found to be

$$h_\xi = 0, \tag{65}$$

$$h_\eta \sim (r)^{1/2} \cos (\phi/2) \cdot 2i(\epsilon/\mu)^{1/2} \sec^2 \alpha_0 \frac{\partial v}{\partial \xi_0}, \tag{66}$$

$$h_z \sim (r)^{1/2} \sin (\phi/2) \cdot 2(\epsilon/\mu)^{1/2} \tan \alpha_0 \sec \alpha_0 \frac{\partial v}{\partial \xi_0}. \tag{67}$$

It is evident from (66) that our edge condition (8) is satisfied. We remark also that the edge behavior given by (62), (63), (64) verifies that obtained by Toraldo di Francia

[1] in an approximate treatment of diffraction by a unidirectionally conducting small circular disc. The field behavior in the vicinity of a diffracting edge is, as anticipated, independent of the shape of the screen.

5. Summary. The results of the foregoing analysis may be summarized in the form of a theorem:

Theorem. Let a dipole field $\mathbf{e}_0$, $\mathbf{h}_0$ derived from the Hertz vector

$$(0, 0, \Pi_z), \quad \text{with} \quad \Pi_z = (e^{-ikR}/kR),$$

be incident upon a screen

$$x \geq 0, \qquad -\infty < y < \infty, \qquad z = 0$$

which has infinite conductivity in the ξ-direction, where

$$\xi = x \cos \alpha_0 - y \sin \alpha_0 \qquad (0 < \alpha_0 < \pi/2),$$

and is perfectly insulating in the direction η normal to ξ:

$$\eta = x \sin \alpha_0 + y \cos \alpha_0 .$$

If the resulting scattered field $\mathbf{e}$, $\mathbf{h}$ is required to satisfy Maxwell's equations, to be outgoing at infinity, and to meet the conditions (boundary condition, two jump conditions and an edge condition) of unidirectionality:

BC:	$e_\xi = 0$,	on the screen
JC^1:	$[h_\xi] = 0$,	across the screen
JC^2:	$[e_\eta] = 0$,	across the screen
EC:	$[h_\eta] = 0$,	at the edge of the screen,

and also to be integrable at the edge of the screen, then the total (vector) field

$$\mathcal{E} = \mathbf{e} + \mathbf{e}_0 , \qquad \mathcal{H} = \mathbf{h} + \mathbf{h}_0$$

is given uniquely by

$$\mathcal{E} = \{\Phi_{\xi z_0} + \Psi^{(1)}, \Phi_{\eta z_0} + \Psi^{(2)}, \Phi_{zz_0} + k^2\Pi_z + \Psi^{(3)}\},$$

$$\mathcal{H} = i\omega\epsilon\left\{\frac{\partial \Pi_z}{\partial \eta}, -\frac{\partial \Pi_z}{\partial \xi} + b^{(\eta)}, b^{(z)}\right\},$$

where the components of the vectors $\mathcal{E}$, $\mathcal{H}$ are in the ξ-, η- and z-directions, and where Φ is Macdonald's classical solution, vanishing on the screen, and given by

$$\Phi = I_R - I_S ,$$

with

$$I_R = \frac{i}{2}\int_{-\mu_R}^{\infty} H_1^{(2)}(kR \cosh \mu)\, d\mu,$$

$$I_S = \frac{i}{2}\int_{-\mu_S}^{\infty} H_1^{(2)}(kS \cosh \mu)\, d\mu,$$

and

$$R = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2},$$
$$S = [(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2]^{1/2},$$
$$\mu_R = \sinh^{-1} \{(2/R)(rr_0)^{1/2} \cos [(\phi - \phi_0)/2]\},$$
$$\mu_S = \sinh^{-1} \{(2/S)(rr_0)^{1/2} \cos [(\phi + \phi_0)/2]\},$$
$$x = r \cos \phi, \qquad z = r \sin \phi, \qquad x_0 = r_0 \cos \phi_0, \qquad z_0 = r_0 \sin \phi_0 .$$

The functions $\Psi^{(1)}$, $\Psi^{(2)}$, $\Psi^{(3)}$ are defined by

$$\Psi^{(1)} = r^{-1/2} \sin (\phi/2)\Big\{\pi ik \cos \alpha_0 \sin (\phi_0/2) \exp (-ik\xi_0) \cdot \frac{\partial}{\partial z_0} \int_{-\infty}^{\xi_0} \frac{(r + r_0')}{(r_0')^{1/2}} \frac{H_1^{(2)}(k\rho')}{\rho'} \exp (ik\xi_0')\, d\xi_0'\Big\},$$

$$\Psi^{(2)} = e^{-ik\xi} \int_{-\infty}^{\xi} \exp (ik\xi') \left.\frac{\partial \Psi^{(1)}}{\partial \eta}\right|_{\xi=\xi'} d\xi',$$

$$\Psi^{(3)} = e^{-ik\xi} \int_{-\infty}^{\xi} \exp (ik\xi') \left.\frac{\partial \Psi^{(1)}}{\partial z}\right|_{\xi=\xi'} d\xi',$$

where $\rho = [(y - y_0)^2 + (r + r_0)^2]^{1/2}$, and the functions $b^{(\eta)}$, $b^{(z)}$ are given by

$$b^{(\eta)} = \int_{-\infty}^{\xi} (F_{zz_0} + \Psi^{(3)}) \Big|_{\xi=\xi'} d\xi'$$

$$b^{(z)} = -\int_{-\infty}^{\xi} (F_{\eta z_0} + \Psi^{(2)}) \Big|_{\xi=\xi'} d\xi'$$

with $F = -\Pi_z + \Phi$.

Some of the field components are singular as the edge of the screen is approached, but these singularities are of the physically admissible $0\ (r^{-1/2})$ type, as may be seen from Eqs. (62) through (67) above. It is especially to be noticed, among our results on edge behavior, that the y-component of the electric field vanishes along the edge of the screen; this follows either from (62), (63) or from (38). We conclude that a suitable edge condition for our problem is

$$e_y = 0,$$

at the edge of the screen, exactly as for a perfectly conducting screen of the same geometry. The use of the corresponding condition, namely that the field component tangential to the rim of the screen be required to vanish, is therefore suggested for further investigations of unidirectional screens. This condition is met by the approximate solution of Toraldo di Francia [1].

We remark that we have given the highest order terms in r, in discussing edge behavior: it is assumed that r_0 is finite. If however we let $r_0 \rightarrow \infty$, the character of the results for field behavior near the edge simplifies further. This type of approximation was given by Senior [4] in discussing edge behavior for the incidence of a dipole field on a perfectly conducting screen.

We conclude by verifying our theorem. Let us write Maxwell's equations in the form

$$(\Delta_{\xi\eta z} + k^2)\mathcal{H} = 0,$$

$$\nabla \cdot \mathcal{H} = 0,$$

$$i\omega\epsilon\mathcal{E} = \nabla \times \mathcal{H}.$$

Notice now that $\Psi^{(1)}$ is a wave function, and that $\Psi^{(2)}$, $\Psi^{(3)}$ are therefore wave functions. It is then clear that $b^{(\eta)}$, $b^{(z)}$ are wave functions. Remarking [see (55), (59)] that

$$\frac{\partial \Psi^{(1)}}{\partial \xi} + \frac{\partial \Psi^{(2)}}{\partial \eta} + \frac{\partial \Psi^{(3)}}{\partial z} = k^2 F_z.$$

we find that $(\mathcal{E}, \mathcal{H})$ is an electromagnetic field. The boundary condition

$$e_\xi = -e_{0\xi}$$

is satisfied; and the first jump condition is obviously met, since $h_\xi = 0$. We have pointed out in giving $\Psi^{(2)}$ that it is an even function of z; and it is clear that $e\eta$ is continuous across the screen, since $\Phi_{\eta z_0}$ is. The edge condition $[h_\eta] = 0$ at the edge of the screen follows from the edge behavior of $h\eta$.

The author wishes to express his indebtedness and gratitude, for suggesting the problem here treated as well as for many valuable discussions, to Professor S. N. Karp.

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BOOK REVIEWS

Switching circuits and logical design. By Samuel H. Caldwell. John Wiley & Sons, Inc., New York, 1958. xviii + 686 pp. $14.00.

This book is needed by every serious student of the logical design of switching circuits. The author has performed a much needed service by gathering most of the significant material on this subject into a single volume, reworking it into a coherent and well-written textbook.

The jacket blurb about Professor Caldwell is unnecessarily modest about his qualifications to have written this book. The two most important advances in the field of logical design were papers by Shannon and by Huffman, and each of these papers was written as an M.I.T. thesis for which Caldwell was a thesis adviser. Understandably he gives much space in his book to exposition of the ideas of these and later publications by Shannon and Huffman, and of significant unpublished material by these two men. The book also contains unpublished material original with Caldwell, and various items due to other M.I.T. men. The author takes so much advantage of the excellent resources of M.I.T. that he errs somewhat on the side of provincialism, tending to ignore some of the important papers written elsewhere, such as Gilbert's work on frontal switching functions, and Muller's work on complexity in electronic switching circuits. He completely omits all work on logical design done outside the English-speaking world. The books by Gavrilov, Higonnet and Grea, and Plechl are never mentioned, and the papers by Lunts, Povarov, Shestakov, Cardot, and others are similarly ignored.

But many useful methods are treated, some of which had not been published or widely known before. This book gives a much more full and thorough coverage of the subject of logical design than any of the other books on the field, including those mentioned above, and the English language ones by Keister, Ritchie, and Washburn, by the staff of the computation laboratory, by Richards (all cited in the book under review), by Phister (Logical design of digital computers, Wiley, 1958), and by Culbertson (Mathematics and logic for digital devices, Van Nostrand, 1958).

This book gives an intuitive rather than a formal approach to its subject, and depends on many well explained illustrations and many tricky exercises to appeal to the ingenious puzzle-solver type of student, rather than attempting to prove theorems or set up formidable mathematics. The book frequently goes out to the limits of what is known in a given part of the subject, and explicitly or implicitly states many problems which are of current research interest.

A strong point of this book is the emphasis that it gives to asynchronous circuit design. The other recent books on logical design cover this topic very incompletely, despite its practical importance, and the many interesting theoretical problems which arise in dealing with it.

There are two minor difficulties which may stand in the way of the wide use of this book as a textbook. The first of these is that most of it (except Chapters 9, 14, and 15) is written about relay circuits, but that most electrical engineers who will be teachers or students of courses in this subject are prejudiced against relays, as compared to the much faster, more modern, and more glamorous electronic switching elements. From both the puzzle solving and the mathematical points of view, relay circuits have many interesting properties which are not to be found in the electronic circuits, although much of the skill or knowledge gleaned in either area is applicable in the other. Perhaps the advent of the cryotron (a recent electronic switching element which is functionally equivalent to a relay) will make this prejudice obsolete.

The second of these difficulties is the close knit way in which the book is integrated. This would be an advantage when teaching a two-semester course just like the one Caldwell teaches. However, he has not only failed to supply asterisks to show which topics may be omitted without loss of continuity, but has interwoven these topics closely with the absolutely essential ones. Anyone wishing to teach a shorter course or to make room in a two-semester course for other topics not treated in this book, such as computer programming, system design, or arithmetic units, will have a major editing job in store for himself.

But neither of these difficulties as a textbook will impair the lasting value of this book as a reference book, explaining in clear and heuristic language the main methods of logical design or its current value as a source of orientation about the frontiers of the field, with its frank comments on many of the kinds of problems we do not know how to solve.

EDWARD F. MOORE

(Continued on p. 146)

GENERAL SOLUTION OF REYNOLDS EQUATION FOR A JOURNAL BEARING OF FINITE WIDTH*

BY

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1. Introduction. The governing equation of hydrodynamic lubrication, the Reynolds equation, was established in 1886 [1]. Since then many works on the solutions of the equation have been published. For plane slider bearings, where the film thickness of the lubricating fluid is a linear function of the coordinate variable, solutions for both infinite and finite widths have been obtained. Michell [2] and Muskat et al. [3] gave their solutions for finite width bearing in terms of Bessel functions. In journal bearings where the film thickness is not a linear function of the variables, Sommerfeld [4] showed that for the case of infinitely long bearings the generalized Reynolds equation, a partial differential equation, is reduced to an ordinary differential equation. The solution of the problem can be immediately derived by integrations. However, for journal bearings of finite length, all of these investigators mentioned in their papers that they did not succeed in obtaining the general solution for the finite bearing problem. Michell even stated that the exact solution of Reynolds equation is only possible if the viscosity is constant and the film thickness is a linear function of the variables. Following this line of argument, the subsequent works on the solutions of finite journal bearings** are approximations by means of a wide variety of mathematical techniques. The aim of this paper is to establish an exact and complete solution of the Reynolds equation for finite journal bearings.

In Sec. 2 of this paper the Reynolds equation, by suitable transformations of variables, is changed to a comparatively unknown equation, Heun's equation. In Sec. 3 Heun's equation and its function related to the problems are discussed. The convergency of the Heun function is also studied. Then in Sec. 4 the pressure distributions, as well as the eigenvalues and their eigenfunctions, are established. Finally, the load carrying capacity and the coefficient of friction of the bearing are determine in Secs. 5 and 6 respectively.

2. Governing equation. The governing equation of hydrodynamic lubrication [1] is

$$\frac{\partial}{\partial x}\left[h^3 \frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial z}\left[h^3 \frac{\partial p}{\partial z}\right] = 6\mu U \frac{dh}{\partial x}\,, \tag{2.1}$$

where p, h and μ are the pressure, film thickness and viscosity of the lubricating fluid respectively, and U is the relative velocity of bearing surfaces. In journal bearings x and z are taken along the circumferential and axial directions respectively, and the film thickness may be written as

$$h = c(1 + \eta \cos \theta) \quad \text{and} \quad \eta = \frac{e}{c}\,, \tag{2.2}$$

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*See bibliography cited in [5] and a recent work [6].

where c is the radial clearance and η the attitude, a ratio of eccentricity e to radial clearance c.

Introducing dimensionless quantities

$$\theta = \frac{x}{r}, \qquad Z = \frac{z}{r}, \qquad H = \frac{h}{c} = 1 + \eta \cos \theta, \qquad P = \frac{p}{\rho U^2}, \tag{2.3}$$

with r being the radius of the bearing, Equation (2.1) becomes

$$\frac{\partial}{\partial \theta}\left[(1 + \eta \cos \theta)^3 \frac{\partial P}{\partial \theta}\right] + \frac{\partial}{\partial Z}\left[(1 + \eta \cos \theta)^3 \frac{\partial P}{\partial Z}\right] = \frac{6\nu r}{Uc^2}\frac{dH}{d\theta}. \tag{2.4}$$

The boundary conditions of the problem are

$$P(-\pi, Z) = P(\pi, Z), \tag{2.5}$$

$$\frac{\partial P(-\pi, Z)}{\partial \theta} = \frac{\partial P(\pi, Z)}{\partial \theta}, \tag{2.6}$$

$$P(\theta, -L/2) = 0, \tag{2.7}$$

$$P(\theta, L/2) = 0, \tag{2.8}$$

where $L = l/d$ and l and d are the length and diameter of the bearing. Conditions (2.5) and (2.6) assure the continuity of the pressure and its first derivative along its circumferential direction, while the last two conditions, (2.7) and (2.8), state that the pressure at free ends is atmospheric.

In order to remove the non-homogeneous term in Eq. (2.4), we let

$$P(\theta, Z) = \zeta(\theta, Z) + \xi(\theta), \tag{2.9}$$

where $\xi(\theta)$ satisfies the equation of two dimensional journal bearings of infinite length, i.e.

$$\frac{d}{d\theta}\left[(1 + \eta \cos \theta)^3 \frac{d\xi}{d\theta}\right] = \frac{6\nu r}{Uc^2}\frac{dH}{d\theta} \tag{2.10}$$

and its boundary conditions

$$\begin{aligned} \xi(-\pi) &= \xi(\pi), \\ \frac{d\xi(-\pi)}{d\delta} &= \frac{d\xi(\pi)}{d\theta}. \end{aligned} \tag{2.11}$$

The solution of Eq. (2.10) with the given boundary conditions (2.11) is [4]

$$\xi(\theta) = \frac{6\nu r\eta}{Uc^2(2 + \eta^2)} \frac{(2 + \eta \cos \theta)\sin \theta}{(1 + \eta \cos \theta)^2}. \tag{2.12}$$

The other function $\zeta(\theta, Z)$ satisfies

$$\frac{\partial}{\partial \theta}\left[(1 + \eta \cos \theta)^3 \frac{\partial \zeta}{\partial \theta}\right] + \frac{\partial}{\partial Z}\left[(1 + \eta \cos \theta)^3 \frac{\partial \zeta}{\partial Z}\right] = 0 \tag{2.13}$$

and the boundary conditions

$$\zeta(-\pi, Z) = \zeta(\pi, Z), \tag{2.14}$$

$$\frac{\partial \zeta(-\pi, Z)}{\partial \theta} = \frac{\partial \zeta(\pi, Z)}{\partial \theta}, \tag{2.15}$$

$$\zeta(\theta, -L/2) = -\xi(\theta), \tag{2.16}$$

$$\zeta(\theta, L/2) = -\xi(\theta). \tag{2.17}$$

Separation of variables

$$\zeta(\theta, Z) = \phi(\theta)\cdot\psi(Z) \tag{2.18}$$

gives

$$\frac{d^2\phi}{d\theta^2} - \frac{3\eta \sin\theta}{1 + \eta\cos\theta}\frac{d\phi}{d\theta} + \lambda\phi = 0 \tag{2.19}$$

and

$$\frac{d^2\psi}{dZ^2} - \lambda\psi = 0. \tag{2.20}$$

From Sturm-Liouville theory [7], it can be shown that Eq. (2.19) with the given boundary conditions possesses a sequence of solutions corresponding to a sequence of real, non-negative eigenvalues λ_i .

The solution of Eq. (2.20) is then

$$\begin{aligned} \lambda_i &\neq 0, \qquad \psi_i = A_i \cosh(\lambda_i)^{1/2}Z + B_i \sinh(\lambda_i)^{1/2}Z \\ \lambda_i &= 0, \qquad \psi_0 = A_0 Z + B_0 . \end{aligned} \tag{2.21}$$

Furthermore by the substitution of

$$u = \frac{1 - \cos\theta}{2} = \sin^2\frac{\theta}{2} \tag{2.22}$$

Eq. (2.19) becomes

$$\begin{aligned} u(u-1)\left(u - \frac{1+\eta}{2\eta}\right)\frac{d^2\phi}{du^2} + \left[4u^2 - \left(4 + \frac{1}{2\eta}\right)u + \frac{1+\eta}{4\eta}\right]\frac{d\phi}{du} \\ - \lambda\left(u - \frac{1+\eta}{2\eta}\right)\phi = 0. \end{aligned} \tag{2.23}$$

This is known as Heun's equation [8].

3. Heun's equation. Heun's equation

$$\begin{aligned} z(z-1)(z-a)y'' + [(\alpha + \beta + 1)z^2 - \{\alpha + \beta + 1 - \delta + a(\gamma + \delta)\}z \\ + a\gamma]y' + \alpha\beta(z - q)y = 0, \end{aligned} \tag{3.1}$$

with a, q, α, β, γ and δ being arbitrary constants, is closely related to the algebraic form of Lame's equation (if $\gamma = \delta = \frac{1}{2}$, $\alpha + \beta = \frac{1}{2}$). Clearly with $a = q = 1$ or $a = q = 0$, Eq. (3.1) is reduced to the hypergeometric equation.

With regular singularities at 0, 1, a and ∞, the Riemann's P-function of Eq. (3.1) is

$$P\left\{\begin{matrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & \alpha & z \\ 1-\gamma & 1-\delta & (\gamma+\delta)-(\alpha+\beta) & \beta & \end{matrix}\right\}. \tag{3.2}$$

Therefore the solutions of Eq. (3.1) for γ being non-integer are

$$y = 1 + \sum_1^\infty c_n z^n \tag{3.3}$$

and

$$y = z^{1-\gamma}\left[1 + \sum_1^\infty c_n' z^n\right]. \tag{3.4}$$

The first solution is

$$y = 1 + \sum_1^\infty c_n z^n = F(a, q; \alpha, \beta, \gamma, \delta; z), \tag{3.5}$$

where $F(a, q; \alpha, \beta, \gamma, \delta; z)$ is known as the Heun function with

$$a\gamma c_1 = \alpha\beta q$$

$$2a^2\gamma(\gamma + 1)c_2 = (\alpha\beta q)^2 + \{(\alpha + \beta + 1 - \delta) + (\gamma + \delta)a\}\alpha\beta q - a\gamma\alpha\beta$$

and

$$\begin{aligned} a(n + 1)(\gamma + n)c_{n+1} = {} & [(\alpha + \beta + n - \delta)n + an(\gamma + \delta + n - 1) \\ & + \alpha\beta q]c_n - [(n - 1)(\alpha + \beta + n - 1) + \alpha\beta]c_{n-1} . \end{aligned} \tag{3.6}$$

The other solution is

$$y = z^{1-\gamma}\left[1 + \sum_1^\infty c_n' z^n\right] = z^{1-\gamma}F(a, q_1 ; \alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, \delta; z) \tag{3.7}$$

with

$$q_1 = \left\{q + \frac{1 - \gamma}{\alpha\beta}\left[(\alpha + \beta) + \delta(a - 1) + (1 - \gamma)\right]\right\}\frac{\alpha\beta}{(\alpha - \gamma + 1)(\beta - \gamma + 1)}.$$

Now to establish the convergence of the function (3.5) we consider the series (the Heun function with argument $z = 1$),

$$\sum_{n=0}^\infty C_n \tag{3.8}$$

and the recurrence relation

and

$$\left.\begin{aligned} C_0 = 1, \qquad P_1C_1 = Q_0 \\ P_{n+1}C_{n+1} = Q_nC_n - R_nC_{n-1} , \end{aligned}\right\} \tag{3.9}$$

where

$$\left.\begin{aligned} Q_0 &= \alpha\beta q, \\ P_n &= \frac{a}{n}(\gamma + n - 1), \\ Q_n &= \frac{1}{n^2}\left[(\alpha + \beta + n - \delta)n + an(\gamma + \delta + n - 1) + \alpha\beta q\right], \\ R_n &= \frac{1}{n^2}\left[(n - 1)(\alpha + \beta + n - 1) + \alpha\beta\right]. \end{aligned}\right\} \tag{3.10}$$

Let

$$N_n = \frac{C_n}{C_{n-1}} \tag{3.11}$$

and

$$P = \lim_{n\to\infty} P_n\,, \qquad Q = \lim_{n\to\infty} Q_n\,, \qquad R = \lim_{n\to\infty} R_n$$

then the recurrence relation (3.9) becomes

$$N_n = \frac{R_n}{Q_n - P_{n+1}N_{n+1}}. \tag{3.12}$$

By repeating substitutions, Eq. (3.12) may be written as an infinite continued fraction,

$$N_n = \frac{R_n}{|\,Q_n} - \frac{P_{n+1}R_{n+1}\,|}{|\quad Q_{n+1}} - \frac{P_{n+2}R_{n+2}\,|}{|\quad Q_{n+2}} - \cdots . \tag{3.13}$$

It can be shown [9] that this continued fraction is convergent when the roots $(\rho_1\,,\,\rho_2)$ of the equation

$$P\rho^2 - Q\rho + R = 0 \tag{3.14}$$

are $|\,\rho_1\,| \neq |\,\rho_2\,|$, or $\rho_1 = \rho_2$ and it is divergent when $\rho_1 \neq \rho_2$ but $|\,\rho_1\,| = |\,\rho_2\,|$. Also in the convergent case with $|\,\rho_1\,| < |\,\rho_2\,|$ or $\rho_1 = \rho_2$ we have

$$\lim_{n\to\infty} N_n \to \rho_1\,. \tag{3.15}$$

In the present series $P = a$, $Q = 1 + a$, and $R = 1$, the roots of Eq. (3.14) are 1 and $1/a$. This, in turn, shows that the Heun function (3.5) is convergent for all $|\,z\,| < |\,a\,|$ if $|\,a\,| > 1$ or for all $|\,z\,| < 1$ if $|\,a\,| < 1$ or $a = 1$.

4. Pressure distribution. By comparison of Eqs. (2.23) and (3.1) we get

$$a = \frac{1+\eta}{2\eta}, \qquad q = \frac{1+\eta}{2\eta}, \qquad \alpha, \beta = \frac{1}{2}\,[3 \pm (9 + 4\lambda)^{1/2}]$$
$$\gamma = \frac{1}{2}, \qquad \delta = \frac{1}{2}. \tag{4.1}$$

Since $a > 1$ when $\eta < 1$, the Heun function of the problem is convergent for all θ. And $a = 1$ when $\eta = 1$, the Heun function (or the hypergeometric function) is divergent. However, $\eta = 1$ implies a contact of bearing surfaces. This is no longer a case of hydrodynamic lubrication. Hence the complete solution of Eq. (2.13) is

$$\zeta = \phi_0\psi_0 + \sum_{i=1}^{\infty} [A_i \cosh(\lambda_i)^{1/2}Z + B_i \sinh(\lambda_i)^{1/2}Z](C_iF_i + D_iH_i), \tag{4.2}$$

where

$$\begin{aligned}
F_i &= F\left(\frac{1+\eta}{2\eta}, \frac{1+\eta}{2\eta}; \frac{3 + (9 + 4\lambda_i)^{1/2}}{2}, \frac{3 - (9 + 4\lambda_i)^{1/2}}{2}, \frac{1}{2}, \frac{1}{2}; u\right),\\
H_i &= u^{1/2}F\left(\frac{1+\eta}{2\eta}, q_1; \frac{4 + (9 + 4\lambda_i)^{1/2}}{2}, \frac{4 - (9 + 4\lambda_i)^{1/2}}{2}, \frac{3}{2}, \frac{1}{2}; u\right),\\
q_1 &= \left(\frac{7}{4} - \lambda_i\right)^{-1}\left[\frac{3}{2} + \frac{1+\eta}{2\eta}\left(\frac{1}{4} - \lambda_i\right)\right],\\
\psi_0 &= A_0Z + B_0\,,\\
\phi_0 &= C_0 \int (1 + \eta\cos\theta)^{-3}\, d\theta + D_0
\end{aligned} \tag{4.3}$$

and A_i, B_i, C_i and D_i are constants.

Conditions (2.16) and (2.17) indicate that ζ is an even function of Z, so

$$A_0 = 0; \qquad B_i = 0.$$

And Eqs. (2.12), (2.16) and (2.17) imply that ζ is an odd function of θ. Since

$$u = \sin^2 \frac{\theta}{2}$$

F_i and H_i are, therefore, even and odd functions of θ respectively. This gives $C_i = 0$. From conditions (2.16), (2.17), (2.7) and (2.8) we then get

$$C_0 = D_0 = 0.$$

Also conditions (2.14) and (2.15) require

$$D_i H_i(\pi) = 0,$$

$$C_i F_i'(\pi) = 0.$$

Since $D_i \neq 0$,

$$H_i(\pi) = 0.$$

Now the pressure distribution with A_i to be determined is

$$P(\theta, Z) = H^*(\theta) + \sum_{i=1}^{\infty} A_i \cosh (\lambda_i)^{1/2} Z \cdot H_i(\lambda_i \,;\, \theta), \tag{4.4}$$

where

$$H^*(\theta) = \xi(\theta) = \frac{6\nu r\eta(2 + \eta \cos \theta) \sin \theta}{Uc^2(2 + \eta^2)(1 + \eta \cos \theta)^2} \tag{4.5}$$

and the eigenvalues λ_i are the roots of

$$H(\lambda; \pi) = 0. \tag{4.6}$$

$H_i(\theta)$ are orthogonal with respect to the weight function

$$w(\theta) = (1 + \eta \cos \theta)^3 \tag{4.7}$$

over the interval $(-\pi, \pi)$; that is

$$\int_{-\pi}^{\pi} w(\theta) H_i(\theta) H_j(\theta) \, d\theta = 0 \quad \text{for} \quad i \neq j \tag{4.8}$$

hence

$$A_i = -\frac{\displaystyle\int_{-\pi}^{\pi} w(\theta) H^*(\theta) H_i(\theta) \, d\theta}{\cosh [(\lambda_i)^{1/2} L/2] \displaystyle\int_{-\pi}^{\pi} w(\theta) H_i^2(\theta) \, d\theta}. \tag{4.9}$$

5. Load carrying capacity. Relating the pressure distribution to the load carrying capacity W of the bearing, W is resolved into two components along and perpendicular to the line of centers,

$$W_1 = \rho U^2 r^2 \int_{-L/2}^{L/2} \int_{-\pi}^{\pi} P \cos \theta \, d\theta \, dZ,$$
$$W_2 = \rho U^2 r^2 \int_{-L/2}^{L/2} \int_{-\pi}^{\pi} P \sin \theta \, d\theta \, dZ. \tag{5.1}$$

Due to the fact that P is an odd function of θ, we have

$$W_1 = 0 \tag{5.2}$$

and

$$\frac{W_2}{\rho U^2 r^2} = \frac{12\pi\nu\eta L r}{Uc^2(2 + \eta^2)(1 - \eta^2)^{1/2}} + \sum_{i=1}^{\infty} \frac{2A_i}{(\lambda_i)^{1/2}} \sinh [(\lambda_i)^{1/2} L/2] \int_0^1 H_i(\lambda_i \; ; u) \, du. \tag{5.3}$$

This result indicates that the line of centers is perpendicular to the load line. The same result has been obtained for the infinitely long journal bearing. With A_i always negative in the range $0 \leq \theta \leq \pi$, the last term of Eq. (5.3) is therefore always negative. This implies that the load capacity of finite length bearing is less than that of infinite length bearing. This result has also been found experimentally by electric analogy [10] and theoretically by approximation methods [3, 6].

6. Coefficient of friction. The frictional force acting on a journal bearing may be expressed by

$$F = r^2 \int_{-L/2}^{L/2} \int_{-\pi}^{\pi} \left[\frac{\mu U}{c(1 + \eta \cos \theta)} + \frac{c\rho U^2}{2r} (1 + \eta \cos \theta) \frac{\partial P}{\partial \theta} \right] d\theta \, dZ. \tag{6.1}$$

By Eq. (5.3) we obtain

$$F = \frac{2\pi\mu U r^2 L}{c(1 - \eta^2)^{1/2}} + \frac{c\eta}{2r} W_2 \, . \tag{6.2}$$

The coefficient of friction is defined as the ratio of fractional force to the total load W. And $W_1 = 0$, we have

$$f = \frac{F}{W} = \frac{2\pi\mu U r^2 L}{c(1 - \eta^2)^{1/2} W_2} + \frac{c\eta}{2r}. \tag{6.3}$$

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SHAPE-PRESERVING SOLUTIONS OF THE TIME-DEPENDENT DIFFUSION EQUATION*

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Abstract. Exact solutions to the time-dependent diffusion equation are exhibited which correspond to the diffusion-limited growth of ellipsoidal precipitate particles with constant shape and dimensions proportional to the square root of the time. The asymmetry of the diffusion field in these solutions is consistent with the preservation of the particle's shape during growth even if the diffusivity is anisotropic. Limiting cases for simpler geometries are derived and shown to be in agreement with previously known results for radially symmetric particles and isotropic diffusion. Similar solutions for hyperboloidal surfaces are exhibited and generalizations are considered analogous to those discussed by Danckwerts for one-dimensional diffusion.

I. Introduction. Exact solutions of the time-dependent diffusion equation are known [1–5] which correspond to the growth of radially symmetric particles in one, two, and three dimensions with radius proportional to the square root of the time. The initial density of diffusing material is uniform and the density at the surface of the particle is specified. The rate of growth is directly proportional to the concentration gradient at the surface. It is the purpose of this paper to show that these solutions are special cases of a more general class of shape-preserving solutions appropriate to ellipsoidal and hyperboloidal surfaces. The generalized boundary condition for the rate of growth is that the velocity of motion of the surface along its normal is proportional to the normal component of the concentration gradient. These more general solutions are of interest in connection with studies of diffusion-limited phase transformations for which the interface between the phases is not radially symmetric or for which the diffusivity is anisotropic, or both.

II. General solutions. We shall examine only those solutions of the isotropic time-dependent diffusion equation

$$\frac{\partial \rho}{\partial t} = D\nabla^2 \rho \tag{1}$$

which are functions of the reduced variables

$$\begin{aligned} u &= xt^{-1/2}, \\ v &= yt^{-1/2}, \\ w &= zt^{-1/2}, \end{aligned} \tag{2}$$

where x, y, z are Cartesian coordinates of position and t is the time. In (1) ρ is the density of diffusing material and D an appropriate diffusivity, assumed for the present to be a scalar. Introducing the vector $\mathbf{s} = (u, v, w)$ and the gradient operator $\nabla_s = (\partial/\partial u$

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$\partial/\partial v$, $\partial/\partial w$), we find that (1) becomes

$$-\tfrac{1}{2}\mathbf{s}\cdot\nabla_s\rho = D\nabla_s^2\rho. \tag{3}$$

We now introduce general ellipsoidal coordinates according to the relations [6]

$$u = [(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)/a^2(a^2 - b^2)]^{1/2}$$
$$v = [(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)/b^2(b^2 - a^2)]^{1/2}$$
$$w = \xi_1\xi_2\xi_3/ab \tag{4}$$

with $\xi_1 > a > \xi_2 > b > \xi_3 > 0$. We find that in these coordinates (3) becomes

$$-\frac{1}{2}\sum_i \frac{\xi_i}{h_i^2}\frac{\partial\rho}{\partial\xi_i} = \frac{D}{h_1h_2h_3}\sum_i \frac{\partial}{\partial\xi_i}\left(\frac{h_1h_2h_3}{h_i^2}\frac{\partial\rho}{\partial\xi_i}\right), \tag{5}$$

where

$$h_i = [(\xi_i^2 - \xi_{i+1}^2)(\xi_i^2 - \xi_{i+2}^2)/(\xi_i^2 - a^2)(\xi_i^2 - b^2)]^{1/2}, \tag{6}$$

the variables ξ_1, ξ_2, ξ_3 being placed in cyclic order.

We shall be concerned with those solutions of (5) which are functions of one of the ξ's only, ξ_i. Because $h_1h_2h_3/h_i^2$ equals

$$f(\xi_i) = [(\xi_i^2 - a^2)(\xi_i^2 - b^2)]^{1/2} \tag{7}$$

times a function of ξ_{i+1} and ξ_{i+2}, under these restrictions (5) becomes

$$-\frac{1}{2}\xi_i f(\xi_i)\frac{d\rho}{d\xi_i} = D\frac{d}{d\xi_i}\left[f(\xi_i)\frac{d\rho}{d\xi_i}\right]. \tag{8}$$

The general solution of (8) is

$$\rho(\xi_i) = c\int_d^{\xi_i}\frac{\exp[-t^2/4D]}{f(t)}\,dt, \tag{9}$$

where c and d are arbitrary constants.

The surface $\xi_1 = \xi_0 > a \geq b \geq 0$ corresponds to the growing ellipsoid in the original x, y, z coordinate system with principal axes of length $2t^{1/2}(\xi_0^2 - a^2)^{1/2}$, $2t^{1/2}(\xi_0^2 - b^2)^{1/2}$, $2t^{1/2}\xi_0$ along the x, y, z axes respectively. Hence the solution of (1) which equals ρ_s on this surface and $(\rho_0 + \rho_s)$ at infinity is from (9)

$$\rho(\xi_1) = \rho_s + \rho_0\{1 - [F(\xi_1)/F(\xi_0)]\}, \tag{10}$$

where

$$F(\xi_1) = \int_{\xi_1}^{\infty}\frac{\exp[-t^2/4D]\,dt}{f(t)} \tag{11}$$

with $f(t)$ defined in (7). At $t = 0$ this corresponds to the uniform density $\rho = \rho_0 + \rho_s$. The rate at which the ellipsoid grows is determined by ξ_0, its shape by the factors $[1 - (a/\xi_0)^2]^{1/2}$, $[1 - (b/\xi_0)^2]^{1/2}$. To determine ξ_0 we shall require that the instantaneous velocity $\mathbf{v}$ in the x, y, z coordinate system of a point (u_0, v_0, w_0) on the surface $\xi_i = \xi_0$ shall have a component along the normal $\boldsymbol{\xi}_i$ to the surface which is proportional to the normal component of the gradient of ρ:

$$(\rho_c - \rho_s)(\mathbf{v}\cdot\boldsymbol{\xi}_i) = D(\boldsymbol{\xi}_i\cdot\nabla\rho). \tag{12}$$

But from (2) we find

$$\mathbf{r} = (x, y, z) = t^{1/2}\mathbf{s}, \tag{13}$$

$$\mathbf{v} = \tfrac{1}{2}t^{-1/2}\mathbf{s}, \tag{14}$$

$$\nabla\rho = t^{-1/2}\nabla_s\rho. \tag{15}$$

Moreover, if ξ_i is a unit vector we can show from (4) that

$$(\mathbf{s}\cdot\boldsymbol{\xi}_i) = \xi_i/h_i\ . \tag{16}$$

Hence (12) becomes

$$\tfrac{1}{2}(\rho_c - \rho_s)\xi_i = D(\partial\rho/\partial\xi_i) \tag{17}$$

to be satisfied for $\xi_i = \xi_1 = \xi_0$. From (10), therefore, ξ_0 must satisfy

$$\tfrac{1}{2}(\rho_c - \rho_s)\xi_0 F(\xi_0) = -D\rho_0 F'(\xi_0), \tag{18}$$

where $F'(\xi) = (d/d\xi)\ F(\xi)$. This equation has a solution if $\rho_0 < (\rho_c - \rho_s)$.

Physically (12) corresponds to the requirement that the flow of diffusing material to the surface be equal to the rate at which the amount of the new phase must be added locally if the surface is to advance with a velocity $\mathbf{v}$. In (12) ρ_c represents the density of the diffusing material in the precipitate phase. In exhibiting (10) and (18), therefore, we have demonstrated the existence of a solution to (1) which satisfies this condition, with $\mathbf{v}$ given by (14), at every point of an ellipsoidal surface. The shape of the diffusion field for a density that is uniform at $t = 0$ and constant at the surface during growth is therefore consistent with the growth of an ellipsoid of arbitrary but constant shape and dimensions proportional to $t^{1/2}$.*

Another interesting solution is obtained from (9) and (17) if we use $\xi_i = \xi_3$. Surfaces of constant ξ_3 $(0 \le \xi_3 < b \le a)$ are hyperboloids of two sheets, with $\xi_3 = 0$ being the u, v plane. Therefore

$$\rho(\xi_3) = \rho_0 + (\rho_s - \rho_0)[G(\xi_3)/G(\xi_0)] \tag{19}$$

with

$$G(\xi_3) = \int_0^{\xi_3} \frac{\exp\left[-t^2/4D\right]\, dt}{f(t)} \tag{20}$$

is a solution of (1) which equals ρ_0 on the x, y plane and ρ_s on the hyperboloid $\xi_3 = \xi_0 < b \le a$, or

$$\frac{x^2}{a^2 - \xi_0^2} + \frac{y^2}{b^2 - \xi_0^2} - \frac{z^2}{\xi_0^2} = -t. \tag{21}$$

This hyperboloid has a trace on the x, z plane which is a hyperbola with foci at $y = 0$, $z = \pm\, at^{1/2}$ and as asymptotes the lines $y = 0$, $x = \pm\,[(a^2 - \xi_0^2)/\xi_0^2]^{1/2}z$. On the y, z

*This analysis is not appropriate for the description of the growth of finite plates or rods of zero thickness (the limiting ellipsoidal forms), for the velocity of growth of the finite dimensions of the particle is found from (18) to be infinite in this limit. However, our ellipsoidal solutions reduce in this limit to those for planes or cylinders given in Sec. III. Thus the length or breadth of the particle becomes infinite in any finite time, and the thickness grows at a finite rate given by the simpler solutions for one or two dimensions. Such infinite growth velocities obviously make the solutions inadequate representations of any physical situation involving a finite three-dimensional particle, but this is not surprising since we do not expect the boundary conditions used in our analysis to be reasonable when any dimension of the particle is smaller than a few atomic diameters.

plane the foci are at $x = 0$, $z = \pm bt^{1/2}$ and the asymptotes $x = 0$, $y = \pm [(b^2 - \xi_0^2)/\xi_0^2]^{1/2}z$.

If we impose the boundary condition (12) or (17) on the surface $\xi_3 = \xi_0$, we find that ξ_0 must satisfy the equation

$$\tfrac{1}{2}(\rho_c - \rho_s)\xi_0 G(\xi_0) = D(\rho_s - \rho_0)G'(\xi_0), \tag{22}$$

where $G' = (dG/d\xi)$. Therefore (19) and (22) correspond to the diffusion-limited transfer of material between one branch of a dissolving hyperboloidal surface of two sheets and the plane containing the conjugate axes of the hyperboloid. Thus if the density of diffusing material on this plane is maintained at a constant value ρ_0 by an external agency, the hyperboloid retains its shape as it dissolves, preserving its asymptotes and eccentricity while the distance between its foci and the plane increases in proportion to the square root of the time.

A second type of solution of (1) involving hyperboloidal surfaces corresponds to precipitation on such a surface instead of its dissolution as above. In this case we replace $t^{-1/2}$ in (2) by $(\tau - t)^{-1/2}$. The sign of the left-hand side of (3), (5), and (8) is correspondingly changed, as is that of the exponent in (9). The integrals corresponding to (10) and (11) diverge because of the positive sign of the exponential, so that we can not use a solution of this type to describe the dissolution of an ellipsoidal particle, at least not with the boundary condition of a given density of diffusing material at infinity. However, there is no such difficulty with (19) and (20), so that with a positive sign in the exponential this solution corresponds to transfer of material between the x, y plane and the hyperboloid given by (21) with t replaced by $(\tau - t)$. The shape of the hyperboloid is preserved, but at time $t = \tau$ the surface has become a cone with its apex touching the x, y plane. The solution is valid only for $t < \tau$.

III. Limiting forms of the general solutions. Morse and Feshbach [6] have listed prescriptions whereby one can pass from ellipsoidal coordinates to most of the more familiar coordinate systems by appropriate stretching, compressing, and translating. If these or similar prescriptions are applied to the solutions (10) and (19) and the boundary conditions (18) and (22) as well as the coordinates, solutions to (1) appropriate to these simpler geometries are obtained. These agree for radially symmetric particles with the results of Rieck [1], Frank [2], and Zener [3].

As an example, if we set $\xi_1^2 = a^2 + u'^2$, $\xi_2^2 = b^2 + v'^2$, and $\xi_3 = w'$ in (4), place $b = a \sin \varphi$, and then let a become infinite, we obtain $u = u'$, $v = v'$, $w = w'$, or rectangular coordinates. Making these substitutions in (10), setting $\xi_0^2 = a^2 + u_0'^2$, and taking the limit, we obtain an expression of the same form but with $F(\xi_1)$ replaced by

$$F(u_0') = \int_{u_0'}^{\infty} \exp\,[-u^2/4D]\, du. \tag{23}$$

This replacement in (18) also gives us the correct boundary condition. This is the usual solution [5] in one-dimensional diffusion for precipitation from a semi-infinite medium onto the plane $x = u_0't^{1/2}$.

Similar manipulations of (19) and (22) yield a one-dimensional result for the diffusion of matter between the plane $z = w_0't^{1/2}$ and the plane $z = 0$. Finally, if we start from the second type of solution for hyperboloidal surfaces discussed above, we get a similar result for the planes $z = w_0'(\tau - t)^{1/2}$ and $z = 0$, all exponentials in the expressions corresponding to (19) and (22) having positive arguments in this case.

The results obtained for the various coordinate systems and surfaces as limiting forms of the general ellipsoid can all be put in the form of (10) and (18) if the function $f(t)$ appearing in $F(\xi)$ is defined appropriately. The proper definitions are listed in Table I, along with values of a, b, and c which when substituted into the equation

$$\frac{x^2}{\xi_1^2 - a^2} + \frac{y^2}{\xi_1^2 - b^2} + \frac{z^2}{\xi_1^2 - c^2} = t \tag{24}$$

define the surface over which ξ_1 and $\rho(\xi_1)$ are constant. We restrict ξ_1 to values greater than a, b and c when these numbers are finite. An infinite value for a, b, or c simply deletes the corresponding term from (24).

TABLE I.

Ellipsoidal solutions in special coordinate systems.

Coordinate System	Surface Parameters*			$f(t)$†
	a	b	c	
Ellipsoidal	a	b	0	$[(t^2 - a^2)(t^2 - b^2)]^{1/2}$
Rectangular	0	∞	∞	1
Oblate spheroidal	a	0	0	$t[t^2 - a^2]^{1/2}$
Prolate spheroidal	a	a	0	$[t^2 - a^2]$
Elliptic cylinder	a	0	∞	$[t^2 - a^2]^{1/2}$
Circular cylinder	0	0	∞	t
Spherical	0	0	0	t^2

*See Eq. (24) of text.
†See Eqs. (10), (11), and (18) of text.

TABLE II.

Hyperboloidal solutions in special coordinate systems.

Coordinate System	Surface Parameters*		$f(t)$†
	a	b	
Ellipsoidal	a	b	$[(a^2 - t^2)(b^2 - t^2)]^{1/2}$
Rectangular	∞	∞	1
Prolate spheroidal	a	a	$[a^2 - t^2]$
Elliptic cylinder	∞	b	$[b^2 - t^2]^{1/2}$

*See Eq. (21) of text.
†See Eq. (19), (20), and (22) of text.

The hyperboloidal solutions (19) have interesting and distinct limiting forms only for the rectangular, prolate spheroidal, and elliptic cylinder coordinates. These solutions and the boundary condition all have the form of (19) and (22) with $f(t)$ in (20) appropriately defined. The proper definitions are listed in Table II together with values of a and b which when substituted into (21) (with ξ_3 replacing ξ_0) define the surface over which ξ_3 and $\rho(\xi_3)$ are constant. We restrict ξ_3 to values between zero and the lesser of a and b.

If the argument of the exponential in (20) is made positive and t replaced by $(\tau - t)$ in (21), then (19), (21), (22) and Table II give the hyperboloidal solutions for which the scale of the geometry is shrinking instead of expanding.

Other types of coordinate systems may be obtained from the ellipsoidal coordinates, but either the limiting surface of interest is identical with one we have already listed (for conical coordinates, for example, the interesting surfaces are spheres) or the limit of (10) or (19) is indeterminate (for paraboloidal and similar systems). In the latter case we can show directly that for these systems (5) does not separate in the sense that if ρ is a function of only one coordinate the other coordinates disappear from the equation.

Evaluation of the integrals appearing in our solutions can not be done simply, although those for rectangular and spherical coordinates can be expressed in terms of tabulated error functions. Frank [2] has given tables for the evaluation of (11) for spherical and circular cylinder coordinates. Since $t^2 > a^2 > b^2$ in integrals of type (11), the more complicated integrals can be evaluated by expanding the factor multiplying the exponential in a power series in $(1/t)$ and tabulating integrals of the form $\int_x^\infty t^{-n}$ $\exp(-t^2)dt$. In integrals of type (20), $t^2 < b^2 < a^2$, so that use of a power series in t is appropriate. Also, if $\xi/D^{\frac{1}{2}}$ is sufficiently small, the exponential factor in the integrand can be replaced by unity. The resulting integrals are then integrable in terms of simple functions, the ellipsoidal result becoming an elliptic integral. This approximate procedure is not satisfactory for cylindrical and rectangular coordinates, since (11) diverges in these cases if the exponential is omitted.

In numerical evaluations of these solutions it is convenient to note that the shape of the surface on which the boundary condition is satisfied is fixed by the parameters a^2/ξ_0^2 and b^2/ξ_0^2. With these chosen, the boundary condition (18) or (22) is then an equation for ξ_0, which determines the rate of growth of the surface.

IV. Generalizations. It is possible to generalize our solutions to considerably more complicated situations analogous to those for a plane interface discussed by Danckwerts [4] and Crank [5]. Thus if the diffusion constant has a discontinuity at a concentration ρ_c, we can use one expression of the form (9) in the region $\rho < \rho_c$, another for $\rho > \rho_c$. Conditions of continuity, the shape of the precipitate surface, a growth condition like (17) at the surface, and prescribed values of ρ at infinity, the discontinuity, and the surface then may be used to determine the parameters and integration constants in the solutions in the two regions. Alternatively, we can specify the normal velocity of two confocal surfaces in terms of the rate at which the diffusing material is transported across each surface, in analogy with Danckwerts' "Class A" problems [4, 5]. If we do not relate the surface velocity to the rate of transport but instead determine our solutions by specifying densities at the surfaces, we have the generalization of "Class B" problems. Finally, we can treat the problem in which the growth of the new phase depends on the diffusion of two or more substances. This was solved by Frank for radial symmetry [2].

V. Anisotropic diffusivity. We shall now show that the preceding solutions are also appropriate when the diffusivity is anisotropic. Choosing coordinate axes such that the diffusivity tensor is in diagonal form, we have in place of (1).

$$\frac{\partial \rho}{\partial t} = D_1 \frac{\partial^2 \rho}{\partial x^2} + D_2 \frac{\partial^2 \rho}{\partial y^2} + D_3 \frac{\partial^2 \rho}{\partial z^2}. \tag{25}$$

The transformation

$$\begin{aligned} x' &= x/D_1^{1/2}, \\ y' &= y/D_2^{1/2}, \\ z' &= z/D_3^{1/2}, \end{aligned} \tag{26}$$

then brings (25) to the form of (1) in terms of the primed coordinates, with $D = 1$.

An ellipsoidal or hyperboloidal surface centered at the origin is described by the equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx - G = 0, \tag{27}$$

with suitable values of the coefficients. The principal axes of this surface are in general different from those of the diffusivity tensor. The boundary condition for the diffusion-limited growth of this surface is now

$$(\rho_c - \rho_s)(\mathbf{v}\cdot\mathbf{n}) = \mathbf{n}\cdot\mathfrak{D}\nabla\rho \tag{28}$$

in place of (12), $\mathbf{v}$ being as usual the velocity of a point on the surface, $\mathfrak{D}$ the diffusivity tensor

$$\mathfrak{D} = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix} \tag{29}$$

and $\mathbf{n}$ the vector

$$\mathbf{n} = (2Ax + Dy + Fz, 2By + Dx + Ez, 2Cz + Ey + Fx) \tag{30}$$

which is normal to (27) at the point $\mathbf{r} = (x, y, z)$. The length of $\mathbf{n}$ is of no concern since (28) is linear in the components of $\mathbf{n}$.

If the surface preserves its shape during growth, $\mathbf{v}$ is proportional to the position vector $\mathbf{r}$:

$$\mathbf{v} = Kf(t)\mathbf{r}. \tag{31}$$

Substituting (29), (30), and (31) into (28) and introducing the change of variable (26), we obtain the boundary condition appropriate to shape-conserving growth expressed in the primed coordinates,

$$(\rho_c - \rho_s)(\mathbf{v}'\cdot\mathbf{n}') = (\mathbf{n}'\cdot\nabla'\rho). \tag{32}$$

Here n' is the vector

$$\begin{aligned} \mathbf{n}' = [&2AD_1x' + D(D_1D_2)^{1/2}y' + F(D_1D_3)^{1/2}z', \\ &2BD_2y' + D(D_1D_2)^{1/2}x' + E(D_2D_3)^{1/2}z', \\ &2CD_3z' + E(D_2D_3)^{1/2}y' + F(D_1D_3)^{1/2}x'], \end{aligned} \tag{33}$$

which is normal to the surface into which (27) is transformed by (26) in the primed coordinates. Also

$$\mathbf{v}' = Kf(t)\mathbf{r}'. \tag{34}$$

This boundary condition (32) is identical with (12) (with $D = 1$), however, if $\mathbf{v}'$ is identified as the velocity of a point on the surface in the primed coordinates.* Since in this coordinate system the diffusivity is isotropic, the solution of (1) and (32) for constant initial density is given immediately by our earlier results. These determine K and $f(t)$

*Note that $\mathbf{v}'$ is not the vector obtained by substituting from (26) into the components of $\mathbf{v}$ in (31).

($= t^{-1}$) in (34), and these in turn determine $\mathbf{v}$ in (31). Thus on transforming back to the original coordinates we have the solution to the problem of the growing surface with anisotropic diffusion, and we see that it indeed corresponds to the preservation of the shape of the surface during growth.* Surfaces of constant concentration are not confocal with this boundary surface in the original coordinates, although they are in the coordinates for which the diffusivity is isotropic.

VI. Discussion. Of the solutions of the time-dependent diffusion equation which we have exhibited, those pertaining to the growth of ellipsoidal particles are of particular interest in the theory of diffusion-limited precipitation from supersaturated solution. As the author has pointed out elsewhere in a discussion of various aspects of precipitation theory [7], the existence under appropriate conditions of solutions corresponding to the growth of ellipsoidal particles of constant shape indicates that in these situations the diffusion field is not responsible for any change that may actually occur in the particles' shape. A change should be attributed to some other cause, therefore. This result contrasts with views that have been widely held in the absence of accurate solutions of the diffusion equation for asymmetric particles or an anisotropic diffusivity. Of course, one can not use the diffusion equation or simple boundary conditions to describe the initial stages of formation of an actual precipitate particle, and our solutions are exact for a uniform initial solute distribution only under the physically artificial restriction that the particles have zero dimensions at $t = 0$. Our solutions are, however, useful in approximating real diffusion-limited growth when the particles have reached a size that permits the neglect of atomic fluctuations and other complications of the nucleation process and when transient effects associated with finite initial dimensions of the particles have substantially vanished.

The usefulness of these solutions for asymmetric surfaces is broadened by their applicability to problems for which the diffusivity is anisotropic. Thus the growth of a radially symmetric particle with anisotropic diffusion is mathematically equivalent to growth of an asymmetric particle with isotropic diffusion. Indeed, the general problem of an arbitrarily oriented ellipsoidal particle of zero initial dimensions with anisotropic diffusion can be solved exactly by means of this transformation, as we have shown. In all cases we find that the surface preserves its shape as it grows, the rate of transport of material across each point of the surface sufficing to give a local surface growth velocity appropriate to this result.

The solutions for hyperboloidal surfaces have no immediate practical application

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*This result may appear at first sight to be incorrect in the limiting case in which one or more components of the diffusivity tensor are identically zero. Thus if one interprets the vanishing of D_1 and D_2 in (29) as requiring that no transport occur except in the z-direction, one must conclude that a growing particle extends its dimensions only parallel to the z-axis, so that its shape is not preserved. However, a discontinuity in the solute density (or equivalently an infinite concentration gradient) then exists on the cylinder which is parallel to the z-axis and tangent to the particle. On the other hand, the solution of the text predicts in the limit (D_1, $D_2 \to 0$) that, for example, a sphere of zero initial radius grows at a finite rate and remains spherical. Surfaces of constant concentration are spheroids of revolution which are tangent to the sphere and have their major axes along the z-direction. An infinite concentration gradient therefore exists at the circle of tangency, but in this limiting case the product of a vanishing component of diffusivity and an infinite gradient leads to a finite current density. The different predictions for the limiting case therefore depend upon how one deals with an infinite concentration gradient. The result of the text is valid if we associate with an infinite gradient a current density obtained by approaching the limit from finite diffusivities.

so far as the author is aware. They do correspond to the diffusion-limited dissolution of a conical or wedge-shaped surface placed initially with its apex touching a plane on which the density is maintained constant by an external agency. However, our solutions require that the density vary appropriately in the region between the plane and the cone even at $t = 0$, and this condition would be difficult to achieve in practice.

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BOOK REVIEWS

(*Continued from p. 128*)

The gyroscope—theory and applications. By James B. Scarborough. Interscience Publishers, New York, London, 1958. xii + 257 pp. $6.50.

This is an introduction to the mechanics of the gyroscope, written on an elementary level and covering a fair amount of theory, but with emphasis on applications. Part I (theory) contains a chapter on vector calculus, one on the fundamental principles of mechanics and two chapters on the theory of the gyroscope, treating the free gyro, forced precession and the heavy gyro. Part II (applications) is devoted to all kinds of gyros on vehicles: gyro-compass, spherical pendulum, stabilizers of ships and of monorail cars. The last chapter is devoted to astronomical applications.

One of the features of this book is its simplicity of reasoning. With very few exceptions everything follows from two simple sets of equations, (16.4) and (25.2). The treatment of the gyro-compass is particularly elegant, also the representation of the pseudo-regular precession by means of a Taylor expansion. The part of the book dealing with applications is very complete. Some numerical examples are helpful for the appreciation of the importance of various factors. Frequent references to modern developments are particularly valuable.

In some instances, especially in the part of the book devoted to theory, the simplicity mentioned above is carried too far. The definitions of the concepts "gyroscope" (p. 37), "heavy gyro" (p. 65), "spherical pendulum" and "gyroscopic pendulum" (p. 161) are rather vague. The pseudo-regular precession of the heavy gyro is discussed under very restrictive initial conditions (p. 70). A gyro having two degrees of freedom should not be referred to as a gyro of one degree of freedom (p. 143). The stability investigations on pp. 55 and 61 start from incorrect assumptions; the results are either incorrect or mere first approximations. Equations as (22.2) through (22.4) or (27.2) through (27.4) where one side is a scalar, the other one a vector, should be avoided.

Though this list might be continued, the value of the book as a whole is beyond question.

A suggestion concerning terminology: the term "precession" for a certain partial motion of a gyro stems from the verb "to precede" (lat.: praecedere); one should not derive a new verb "to precess" from this noun but rather say that a gyro precedes if it moves in a precession.

H. Ziegler.

A course in multivariate analysis. By M. G. Kendall. Hafner Publishing Co., New York, 1957. 185 pp. $4.50.

Until recently multivariate analysis has not been well covered by the text books in theoretical statistics. At almost the same time three monographs appear now in this field: the one by M. G. Kendall, another called Introduction to multivariate statistical analysis by T. W. Anderson and a third by S. N. Roy: Some aspects of multivariate analysis. The two latter ones are published by John Wiley & Sons.

Dr. Kendall's book is meant to be an introduction to the subject, perhaps especially for a reader interested more in applying these statistical methods than in their mathematical derivation. Indeed, one attractive feature of the book is the emphasis given to model construction and the interpretation of numerical results. The validity of the available techniques is critically examined, e.g. the centroid method in the first chapter, and alternative methods are examined and compared. A great number of detailed numerical examples illustrate the theoretical discussion and add a good deal to the value and usefulness of the book. The exposition is clear and the book reads very well.

The book begins with a brief introduction describing what multivariate analysis is and to what fields it has been applied. The second chapter is devoted to component analysis and contains some elementary but basic results, which are used throughout the rest of the book. A brief discussion of factor analysis follows, and the fourth chapter is concerned with functional relationships with stochastic elements. This topic is often treated in an obscure way in the literature and often causes difficulties to the student

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EIGENOSCILLATIONS OF AN ELASTIC CABLE*

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Summary. The eigenoscillations of a cable, supported at its end points, in a homogeneous gravitational field is investigated under the assumption that the bending stiffness is negligible but the behavior with respect to tension stresses is perfectly elastic. The problem involves two coupled second order equations and one independent second order equation; it is shown to be definitely self-adjoint and an iterative method for its solution is suggested.

In the particular case of a shallow cable, that is with negligible sag, the asymptotic eigenvalues are obtained. It turns out that the gravitational field has such a "stiffening" effect that the eigenvalues related to some oscillation modes may be substantially greater (the lowest of them eightfold) than those given in the classical theory on vibrating strings.

Notations.

A = cross sectional area, L = total length, s = arc length parameter, ρ = mass density, $\mu = \rho A$ = mass per unit length — of the cable when unstrained;

E = Young's modulus;

x, y, z rectilinear coordinates of a typical cable point; the gravity acting in the direction of negative y-axis, the end points of the cable having coordinates $(\pm \frac{1}{2}l, 0, 0)$;

u, v, w displacement components from equilibrium position;

F = tension force acting on the cable;

f = increment of the force due to the motion;

a = length constant determined by Equation (3);

$\epsilon = a\mu g/EA = a\rho g/E$ equilibrium strain at the lowest point;

$\alpha = L/2a$ = characteristic parameter;

ω = circular frequency of eigenoscillations;

$\lambda^* = \omega^2 a/g$, $\lambda = \kappa^2 = \omega^2 L^2/4ag$ — eigenvalue parameters;

$p = s/a$, $q = 2s/L$ dimensionless variables;

A, **B**, **M**, **N**, **S**, **T** matrices.

Derivation of the equations. From Hooke's law follows

$$x_s'^2 + y_s'^2 + z_s'^2 = (1 + F/EA)^2. \tag{1}$$

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**On leave of absence from the Finland Institute of Technology, Helsinki.

Hence the components of the tension force are

$$\frac{Fx'_s}{1 + F/EA}, \qquad \frac{Fy'_s}{1 + F/EA}, \qquad \frac{Fz'_s}{1 + F/EA}.$$

The influence of the flexural rigidity being neglected, Newton's law gives

$$\begin{cases} \left(\dfrac{Fx'_s}{1 + F/EA}\right)'_s = \mu x''_{tt}, \\ \left(\dfrac{Fy'_s}{1 + F/EA}\right)'_s = \mu(y''_{tt} + g), \\ \left(\dfrac{Fz'_s}{1 + F/EA}\right)'_s = \mu z''_{tt}. \end{cases} \tag{2}$$

If time derivatives are taken to be identically zero, then Eqs. (1) and (2) give rise to the stationary equilibrium solution. Using boundary conditions and choosing the origin of s at the lowest point one obtains

$$\begin{cases} x_0 = a \log [s/a + (1 + s^2/a^2)^{1/2}] + \epsilon s, \\ y_0 = (a^2 + s^2)^{1/2} + \epsilon s^2/2a + K, \\ z_0 = 0 \\ F_0 = \mu g(a^2 + s^2)^{1/2}. \end{cases}$$

The integration constants a and K in these expressions are determined by the conditions

$$\log [L/2a + (1 + L^2/4a^2)^{1/2}] + L\rho g/2E = l/2a, \tag{3}$$
$$(1 + L^2/4a^2)^{1/2} + L^2\rho g/8aE + K/a = 0.$$

After introducing the small oscillation displacements by

$$\begin{cases} x(s, t) = x_0(s) + u(s, t) \\ y(s, t) = y_0(s) + v(s, t) \\ z(s, t) = z_0(s) + w(s, t) \\ F(s, t) = F_0(s) + f(s, t) \end{cases}$$

and substituting these expressions in (1) and (2) the linearization is performed:

$$\frac{u'_s + sv'_s/a}{(1 + s^2/a^2)^{1/2}} = \frac{f}{EA}$$

$$\left(\frac{f + \epsilon EA(1 + s^2/a^2)u'_s}{(1 + s^2/a^2)^{1/2}[1 + \epsilon(1 + s^2/a^2)^{1/2}]}\right)'_s = \mu u''_{tt},$$

$$\left(\frac{sf/a + \epsilon EA(1 + s^2/a^2)v'_s}{(1 + s^2/a^2)^{1/2}[1 + \epsilon(1 + s^2/a^2)^{1/2}]}\right)'_s = \mu v''_{tt},$$

$$\left(\frac{\epsilon EA(1 + s^2/a^2)w'_s}{(1 + s^2/a^2)^{1/2}[1 + \epsilon(1 + s^2/a^2)^{1/2}]}\right)'_s = \mu w''_{tt}.$$

When f is eliminated, the dimensionless variable $p = s/a$ introduced, and the variables separated by $u(s, t) = u(s) \cdot e^{i\omega t}$, etc., then the following system of ordinary differential equations is obtained:

$$\begin{cases} \left(\dfrac{u' + pv' + \epsilon(1 + p^2)^{3/2}u'}{\epsilon(1 + p^2)[1 + \epsilon(1 + p^2)^{1/2}]}\right)' + \lambda^* u = 0, \\ \left(\dfrac{pu' + p^2v' + \epsilon(1 + p^2)^{3/2}v'}{\epsilon(1 + p^2)[1 + \epsilon(1 + p^2)^{1/2}]}\right)' + \lambda^* v = 0, \end{cases} \tag{4}$$

$$\left(\frac{w'}{(1 + p^2)^{-1/2} + \epsilon}\right)' + \lambda^* w = 0. \tag{5}$$

Two first equations apparently determine the modes and frequencies of the eigen-oscillations in the xy-plane, whereas the third equation applies to the perpendicular oscillations, completely independent of the first ones. In the following the main attention will be given to the first oscillations.

The self-adjointness of the problem. Since the eigenvalue problem, consisting of the two second order differential equations (4) and boundary conditions

$$u(\pm\alpha) = v(\pm\alpha) = 0, \qquad (\alpha = L/2a)$$

may be regarded as arising from a regular variational problem, it is self-adjoint. The exact definition of a definitely self-adjoint problem, due to Bliss [1, 2], requires that if the problem is written in the form

$$\begin{cases} \dfrac{d}{dp}\mathbf{u} = \mathbf{Au} + \lambda\mathbf{Bu}, \\ \mathbf{Mu}(a) + \mathbf{Nu}(b) = 0, \end{cases}$$

where $\mathbf{u}$ is an n-vector, $\mathbf{A}$ and $\mathbf{B}$ are $(n \times n)$ matrices and $\mathbf{M}$ and $\mathbf{N}$ constant $(n \times n)$ matrices with $(\mathbf{M}, \mathbf{N})$ being of rank n, then there exists a non-singular matrix $\mathbf{T}$ such that the following conditions are fulfilled:

$$\begin{cases} \dfrac{d}{dp}\mathbf{T} + \mathbf{TA} + \mathbf{A'T} = 0, \\ \mathbf{TB} + \mathbf{B'T} = 0, \end{cases} \tag{6}$$

$$\mathbf{MT}^{-1}(a)\mathbf{M}' = \mathbf{NT}^{-1}(b)\mathbf{N}', \tag{7}$$

(here the prime stands for the transposition of the matrix in question) and that the matrix

$$\mathbf{S} = \mathbf{T'B}$$

is symmetric and definite or semi-definite.

In the present case, solve the Eqs. (4) with respect to u'' and v'' and regard u, v, u', and v' as four elements of $\mathbf{u}$. One finds then that the coefficient matrices are

$$\mathbf{A} = \frac{1}{\epsilon(1 + p^2)^{5/2}[1 + \epsilon(1 + p^2)^{1/2}]} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p + \epsilon p(2 - p^2)(1 + p^2)^{1/2} & p^2 - \epsilon(1 - 2p^2)(1 + p^2)^{1/2} \\ 0 & 0 & -1 - \epsilon(1 - 2p^2)(1 + p^2)^{1/2} & -p - 3\epsilon p(1 + p^2)^{1/2} \end{bmatrix},$$

$$\mathbf{B} = \frac{1}{(1 + p^2)^{3/2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p^2 - \epsilon(1 + p^2)^{3/2} & p & 0 & 0 \\ p & -1 - \epsilon(1 + p^2)^{3/2} & 0 & 0 \end{bmatrix},$$

$$(\mathbf{M}, \mathbf{N}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The solution for (6), (7) is now

$$\mathbf{T} = \frac{1}{\epsilon(1 + p^2)[1 + \epsilon(1 + p^2)^{1/2}]} \cdot \begin{bmatrix} 0 & 0 & 1 + \epsilon(1 + p^2)^{3/2} & +p \\ 0 & 0 & +p & p^2 + \epsilon(1 + p^2)^{3/2} \\ -1 - \epsilon(1 + p^2)^{3/2} & -p & 0 & 0 \\ -p & -p^2 - \epsilon(1 + p^2)^{3/2} & 0 & 0 \end{bmatrix}$$

and the matrix $\mathbf{S}$ is semi definite:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The definitely self-adjoint character of the problem thus being proved, a number of properties associated with such problems may be applied for the present problem: for example, the existence of a countably infinite number of eigensolutions, the expansion theorem, etc. The orthogonality of two independent eigensolutions (u_1, v_1), (u_2, v_2) now takes the form

$$\int_{-\alpha}^{+\alpha} (u_1 u_2 + v_1 v_2)\, dp = 0.$$

Furthermore, an iterative method for the determination of the eigensolutions is applicable and proved to be convergent [3]. The sequence of consecutive iterates is obtained directly from (4):

$$\begin{cases} u_{i+1} = \displaystyle\int \left\{ \left(-\frac{p^2}{(1 + p^2)^{3/2}} - \epsilon \right) \int u_i\, dp + \frac{p}{(1 + p^2)^{3/2}} \int v_i\, dp \right\} dp, \\ v_{i+1} = \displaystyle\int \left\{ \frac{p}{(1 + p^2)^{3/2}} \int u_i\, dp + \left(-\frac{1}{(1 + p^2)^{3/2}} - \epsilon \right) \int v_i\, dp \right\} dp; \end{cases}$$

the integration constants must be chosen to make the functions satisfy the boundary conditions.

Finally, it may be seen directly from the form of the Eq. (5) that the problem for the determination of the lateral oscillations is definitely self-adjoint also and hence subject to similar properties and procedures as those mentioned above.

Case of a shallow cable. The general solution of the problem above depends essentially on two dimensionless parameters only, the relative cable length $\alpha = L/2a$ and the minimum strain ϵ. The latter is always smaller than the ratio between the yield stress and Young's modulus and hence small compared with 1. In the case of a shallow cable the first parameter also is small compared with 1, since a is approximately equal to the smallest radius of curvature. For a closer study of this particular case we replace the variable p by the new variable

$$q = 2s/L = p/\alpha$$

in order to have fixed boundary values $q = \pm 1$. The Eqs. (4) and (5) thus obtain the form

$$\begin{cases} \left(\dfrac{u' + \alpha q v' + \epsilon(1 + \alpha^2 q^2)^{3/2} u'}{\epsilon(1 + \alpha^2 q^2)[1 + \epsilon(1 + \alpha^2 q^2)^{1/2}]}\right)' + \lambda u = 0, \\ \left(\dfrac{\alpha q u' + \alpha^2 q^2 v' + \epsilon(1 + \alpha^2 q^2)^{3/2} v'}{\epsilon(1 + \alpha^2 q^2)[1 + \epsilon(1 + \alpha^2 q^2)^{1/2}]}\right)' + \lambda v = 0, \end{cases} \tag{8}$$

$$\left(\frac{w'}{(1 + \alpha^2 q^2)^{-1/2} + \epsilon}\right)' + \lambda w = 0, \tag{9}$$

where the prime now indicates differentiation with respect to q. The pertinent boundary conditions are

$$u(\pm 1) = v(\pm 1) = w(\pm 1) = 0. \tag{10}$$

After noticing that the last equation apparently has the asymptotic eigenvalues

$$\lambda \sim (n\pi/2)^2, \qquad n = 1, 2, \cdots,$$

for α and ϵ tending to zero, we will leave this equation aside and study the problem (8), (10) for small values of α and ϵ. By letting, separately, $\alpha \to 0$ one arrives at the equations

$$\begin{cases} u'' + \epsilon\lambda u = 0, \\ v'' + (1 + \epsilon)\lambda v = 0, \end{cases}$$

which are not coupled and give the well-known eigenfrequencies of longitudinal and transversal oscillations. Since there is, for a small ϵ, a wide gap between the smallest eigenvalues $\pi^2/4\epsilon$ and $\sim \pi^2/4$ of both groups, it is of interest to find out what are actually the smallest eigenvalues, say when $\epsilon\lambda$ may be regarded to be still essentially smaller than 1. To this end observe, that the Eqs. (8) are equivalent with the following two equations, where C_1 and C_2 are arbitrary integration constants:

$$u' + \alpha q v' + \epsilon\lambda\left[\int_0^q u\, dq + \alpha q \int_0^q v\, dq\right] + \epsilon C_1 + \epsilon C_2 \alpha q = 0, \tag{11}$$

$$\frac{\alpha q u' - v'}{\epsilon + (1 + \alpha^2 q^2)^{-1/2}} + \lambda\left[\alpha q \int_0^q u\, dq - \int_0^q v\, dq\right] + C_1 \alpha q - C_2 = 0. \tag{12}$$

Then, considering first the solutions of group I, that is, those for which u is odd and v is even, one immediately finds that $C_2 = 0$. Assume that λ remains bounded for α and ϵ tending to zero and, since both integral terms of Eq. (11) have ϵ as a factor, neglect these terms besides two first terms:

$$u' + \alpha q v' + \epsilon C_1 = 0. \tag{13}$$

Further, substitute u from this equation into the Eq. (12) and neglect all terms which are not of the lowest order in α or ϵ:

$$v' + \lambda \int_0^q v\, dq - C_1 \alpha q = 0.$$

All even solutions of this equation which satisfy the condition $v(1) = 0$, are, up to a constant factor, of the form

$$v = \cos \kappa q - \cos \kappa,$$

whereby C_1 and λ are

$$C_1 = -\kappa^2 \cos \kappa/\alpha, \qquad \lambda = \kappa^2.$$

The substitution into (13) gives odd solutions

$$u = \alpha[\sin \kappa q/\kappa - q \cos \kappa q + \epsilon \kappa^2 q \cos \kappa/\alpha^2].$$

From the condition $u(1) = 0$, κ is determined as a solution of the characteristic equation

$$\tan \kappa/\kappa = 1 - \kappa^2 \epsilon/\alpha^2.$$

Hence, the roots of this equation squared give the eigenvalues.

Quite similarly one finds that the asymptotic solutions of group II, whose u is even and v odd, are

$$\begin{cases} \lambda = n^2\pi^2, \\ u = -(\alpha/n\pi)(\cos n\pi + \cos n\pi q + n\pi q \sin n\pi q), \\ v = \sin n\pi q. \end{cases}$$

Hence in group I the solutions, the eigenvalues as well as the modes of eigenfunctions, depend on the ratio ϵ/α^2, whereas in group II the solutions are independent of this ratio. The dependence is illustrated in Fig 1, where the solid curves give some lowest eigenvalues from the group I, the broken lines eigenvalues from group II. In particular, for $\alpha^2/\epsilon = n^2\pi^2$ there exist double eigenvalues $n^2\pi^2$.

Finally, it may be indicated what the physical significance of the ratio ϵ/α^2 is. Expressing ϵ as the ratio σ/E, where σ is the equilibrium stress at the lowest point, and α by

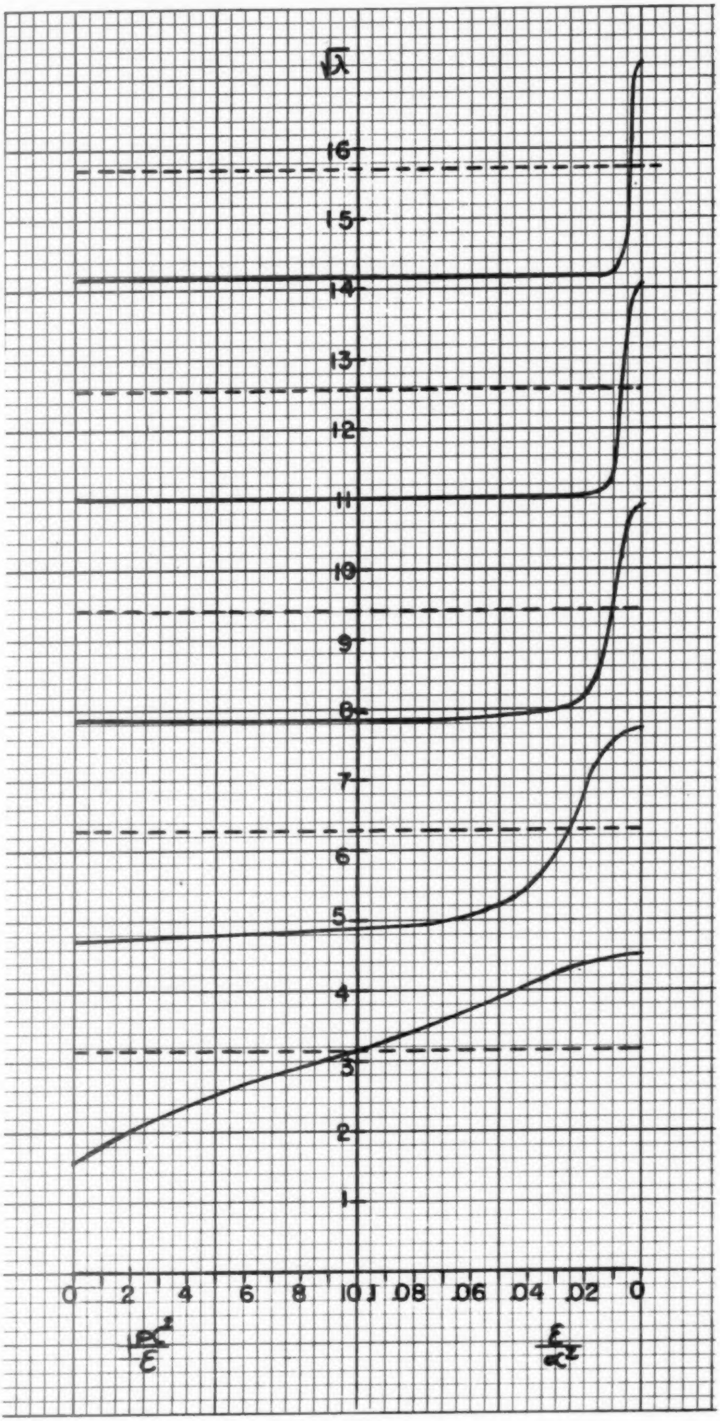
$$\alpha = L/2a = L\rho g/2\epsilon E = L\rho g/2\sigma,$$

we obtain

$$\epsilon/\alpha^2 = 4\sigma^3/L^2\rho^2 g^2 E = 4\cdot(L_0/L)^2,$$

where the characteristic length

$$L_0 = (\sigma/E)^{1/2}\cdot(\sigma/\rho g)$$



$\sqrt{\lambda}$
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0 2 4 6 8 10 .1 .08 .06 .04 .02 0
$\frac{\mu\alpha^2}{\varepsilon}$
$\frac{\varepsilon}{\alpha^2}$

FIG. 1

depends on the material in question and its equilibrium stress. For most materials this length, when computed for stresses reasonably close to their tensile proportionality limit, amounts to several hundreds of feet. For instance, for copper, $\sigma = 30{,}000$ psi, $L_0 = 330$ feet, and for steel, $\sigma = 40{,}000$ psi, $L_0 = 430$ feet. Therefore, the eigenfrequencies of a tightly stressed cable or wire do essentially differ from those corresponding to the well-known eigenvalues $\lambda = (n\pi/2)^2$ only at spans of length comparable with this measure.

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ASYMPTOTIC DEVELOPMENTS FOR A BOUNDARY VALUE PROBLEM CONTAINING A PARAMETER*

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1. Introduction. It is the purpose of this paper to point out a curious type of singularity which can arise in the perturbation of the solutions of boundary value problems containing a parameter. We shall consider two such problems each for functions of x and y continuous in $y \leq 0$, (x, y) bounded, and harmonic in $y < 0$. The two functions satisfy the following sets of boundary condition, K being a positive constant and $g(x)$ a given function:

Problem I $\quad (A_{\text{I}}) \quad u_y(x, 0) = 0 \quad$ for $\quad |x| > 1$

$\qquad (B_{\text{I}}) \quad u_y(x, 0) + Ku(x, 0) = g(x) \quad$ for $\quad |x| < 1,$

Problem II $\quad (A_{\text{II}}) \quad u_y(x, 0) - Ku(x, 0) = 0 \quad$ for $\quad |x| > 1$

$\qquad (B_{\text{II}}) \quad u_y(x, 0) = g(x) \quad$ for $\quad |x| < 1.$

We admit the possibility of the function $g(x)$ depending on K provided it be analytic in K for K sufficiently small. The two problems clearly have much the same character, the one deriving from the other essentially by interchanging the roles of the intervals $|x| < 1$ and $|x| > 1$. In fact it can be shown that they are equivalent, that is solution of the one yields the solution of the other [1]. The two problems indicate strikingly the need for caution in the study of perturbations for despite their apparent similarity we shall find entirely different behavior of the solutions for small K.

Physically (I) is a problem in heat conduction while (II) governs the diffraction of surface water waves by a rigid dock of finite width [2]. Differences in the problem become more apparent when we specify the behavior of the solutions of (I) and (II) for large $x^2 + y^2$. It is easily seen in fact that the following behaviors hold for (I) and (II) respectively, [2].

$$(C_{\text{I}}) \quad u(x, y) - c \log (x^2 + y^2) = 0(x^2 + y^2)^{-1} \quad \text{as} \quad x^2 + y^2 \to \infty, \quad c \text{ constant.}$$

$$(C_{\text{II}}) \quad u(x, y) - Te^{Ky}e^{iKx} = 0\left(\frac{1}{x}\right) \quad \text{as} \quad x \to +\infty, \quad y \text{ bounded}$$

$$u(x, y) - Re^{Ky}e^{iKx} = 0\left(\frac{1}{x}\right) \quad \text{as} \quad x \to -\infty, \quad y \text{ bounded,}$$

for some constants T and R.

If one sets $K = 0$ in (I) and (II) the problems become formally the same namely

$$(A_0) \quad u_y(x, 0) = 0 \qquad \text{for} \quad |x| > 1$$

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$$(B_0) \quad u_y(x, 0) = g(x) \quad \text{for} \quad |x| < 1.$$

This last problem is trivial having as solution the function*,

$$u_0(x, y) = -\frac{1}{\pi} \int_{-1}^{+1} g(t) \log [(x - t)^2 + y^2] \, dt. \tag{1.1}$$

One may ask now whether the solutions of problems (I) and (II) tend to $u_0(x, y)$ as $K \to 0$. A glance at conditions (C_I) and $C_{II})$ leads one to suspect that such is the case for the solution of (I) while for the solution of (II) difficulties might arise, a suspicion which proves correct. We shall see that the solution of (I) is in fact an analytic function, $u(x, y; K)$ for complex K of sufficiently small absolute value, with $u(x, y; 0) = u_0(x, y)$. The solution of (II) is also analytic in K but for a region $0 < |K| < \rho$. It has in fact the rather curious asymptotic expansion,†

$$Ku(x, y; K) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}(x, y) K^m (K \log K)^n \qquad A_{00} \equiv 0. \tag{1.2}$$

Thus the solution of (II) does not possess a limit as $K \to 0$. The apparent physical paradox introduced in the finite dock problem can be removed by more careful attention to physical units but the odd development (1.2) for the solution of (II) remains.

2. Developments for Problem (I). The development of the solution of Problem (I) for small K is elementary. To solve (I) we try the function,

$$u(x, y; K) = -\frac{1}{\pi} \int_{-1}^{+1} f(t; K) \log [(x - t)^2 + y^2] \, dt. \tag{2.1}$$

This function clearly is harmonic in $y < 0$ and satisfies (A_I) and (C_I). By a well known formula, it will satisfy (B_I) provided $f(t; K)$ is a solution of the integral equation,

$$f(x; K) + \frac{K}{\pi} \int_{-1}^{+1} f(t; K) \log |x - t| \, dt = g(x) \qquad |x| < 1. \tag{2.2}$$

The usual method of successive approximations establishes that (2.2) possesses a solution $f(x; K)$ for suitably small K which is analytic in K. Further we have $f(x, 0) = g(x)$. Substitution in (2.1) yields a solution $u(x, y; K)$ of (I) analytic for K sufficiently small and with $u(x, y; 0) = u_0(x, y)$.

3. Developments for Problem (II). An integral representation for Problem (II) can be obtained in the usual manner by introducing the Green's function $G(x, y, t; K)$ defined by,

$$G(x, y, t; K) = -1/2 \log [(x - t)^2 + y^2] + \frac{K}{2} e^{Ky} \int_y^{\infty} e^{-K\eta} \log [(x - t)^2 + \eta^2] \, d\eta + \pi i e^{Ky} e^{iK|x-t|}. \tag{3.1}$$

We have,

$$G_y = KG = -\frac{y}{(x - t)^2 + y^2}.$$

Thus setting,

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*For this we need Hölder continuity of $g(x)$.

†It is to be noted that asymptotic developments of the form (1.2) have been observed before in Problem (II) but in an entirely different connection [3]. The meaning of the asymptotic development is made clear in Sec. 3.

$$u(x, y; K) = -\frac{1}{\pi} \int_{-1}^{+1} f(t; K)G(x, y, t; K)\, dt \tag{3.2}$$

we find, in the same manner as (2.2) was obtained from (2.1),

$$u_y(x, 0; K) = f(x; K) + \frac{K}{\pi} \int_{-1}^{+1} f(t; K)G(x, 0, t; K)\, dt \quad \text{on} \quad |x| < 1$$

while $u(x, y; K)$ satisfies (A_{II}) for any choice of f. It is further easy to show that the function defined by (3.2) is harmonic in $y < 0$, continuous in $y \leq 0$ and satisfies (C_{II}). Thus (3.2) yields a solution of Problem (II) if $f(x; K)$ is a solution of the integral equation,

$$g(x) = f(x; K) + \frac{K}{\pi} \int_{-1}^{+1} f(t; K)G(x, 0, t; K)\, dt \quad \text{on} \quad |x| < 1. \tag{3.3}$$

Note that the function $f(x)$ is related to the solution u by,

$$f(x; K) = u_y(x, 0; K) - Ku(x, 0; K). \tag{3.4}$$

In order to express $u(x, y; K)$ as a function of the parameter K we must so express $G(x, y, t; K)$ Observe first that,

$$\pi i e^{Ky} e^{iK|x-t|} = \sum_{m=0}^{\infty} A_m(x - t, y)K^m,$$

$$\begin{aligned} &Ke^{Ky} \int_y^0 e^{-K\eta} \log [(x - t)^2 + \eta^2]\, d\eta \\ &\quad = Ke^{Ky} \sum_{m=1}^{\infty} \frac{(-1)^n K^n}{n!} \int_y^0 \eta^n \log [(x - t)^2 + \eta^2]\, d\eta = \sum_{m=1}^{\infty} B_m(x, y, t)K^m, \end{aligned} \tag{3.5}$$

the series converging absolutely and uniformly for bounded x, y, t, and $|K|$ The term,

$$I = Ke^{Ky} \int_0^{\infty} e^{-K\eta} \log [(x - t)^2 + \eta^2]\, d\eta \tag{3.6}$$

is more complicated. Note first that,

$$\begin{aligned} I &= e^{Ky} \int_0^{\infty} e^{-\tau} \log \left[(x - t)^2 + \frac{\tau^2}{K^2}\right] d\tau \\ &= -2e^{Ky} \log K + e^{Ky} \int_0^{\infty} e^{-\tau} \log [(K\,|x - t|)^2 + \tau^2]\, d\tau, \end{aligned}$$

hence,

$$\lim_{K \to 0} (I + 2 \log K) = 2 \int_0^{\infty} e^{-t} \log t\, dt = -2\gamma, \tag{3.7}$$

where γ is Euler's constant.

To obtain a more detailed description of I we study it as a function of complex K. The integral (3.6) defining I converges, and yields an analytic function as long as $0 < |K|$, $0 \leq \arg K < \pi/2$. I may be continued analytically to a larger sector by shifting the path of integration. In order to continue $I(K)$ into $\arg K \geq \pi/2$ we shift the path into the sector $\arg \eta < 0$ so that it keeps the points $\pm i\,|x - t|$ always to the right and runs to ∞ along the ray $\arg \eta = -\arg K$, thus keeping $K\eta$ positive for large $|\eta|$.

In particular if I^+ denotes the value of I after one counter-clockwise circuit of the origin we have,

$$I^+ = Ke^{Ky} \int_c e^{-K\eta} \log [(x - t)^2 + \eta^2] \, d\eta + Ke^{Ky} \int_0^\infty e^{-K\eta} (\log [(x - t)^2 + \eta^2])^- \, d\eta,$$

where c denotes a closed path surrounding $\pm i \mid x - t \mid$ in the negative sense and $(\log [(x - t)^2 + \eta^2])^-$ denotes the value of the logarithm after η circles the points $\pm i \mid x - t \mid$ in the negative sense. Hence,

$$\begin{aligned} I^+ - I &= 2Ke^{Ky}\left\{\pi i \int_0^{-i|x-t|} e^{-K\eta} \, d\eta + \pi i \int_0^{i|x-t|} e^{-K\eta} \, d\eta - 2\pi i \int_0^\infty e^{-K\eta} \, d\eta\right\} \\ &= -4\pi i e^{Ky} \cos K \mid x - t \mid. \end{aligned} \tag{3.8}$$

Equations (3.7) and (3.8) together imply that $I(x, y, t)$ has the form,

$$I = \{-2e^{Ky} \cos K \mid x - t \mid\} \log K + \sum_{m=0}^\infty c_m(x, y, t)K^m, \tag{3.9}$$

where by (3.7) $c_0 = -2\gamma$. Combining (3.5) and (3.9) we have finally,

$$\begin{aligned} G(x, y, t; K) &= \{-e^{Ky} \cos K \mid x - t \mid\} \log K + \sum_{m=0}^\infty G_m(x, y, t)K^m \\ &= -\log K \sum_{m=0}^\infty H_m(x, y, t)K^m + \sum_{m=0}^\infty G_m(x, y, t)K^m \end{aligned} \tag{3.10}$$

the series converging for suitably small K. For reference we write down the first few terms:

$$H_0 = -1 \qquad H_1 = -y \qquad H_2 = -\tfrac{1}{2} \mid x - t \mid^2 + \tfrac{1}{2}y^2,$$
$$G_0 = (\pi i - \gamma) - \log [(x - t)^2 + y^2]^{1/2}.$$

For the case $y = 0$ all of the terms in the development may be obtained from formulae given in [4].

With the developments (3.10) at hand we are ready to discuss $u(x, y; K)$. Suppose $g(x) = \sum_{m=0}^\infty g_m(x)K^m$. Then substituting (3.10) in (3.3) we have,

$$\begin{aligned} f(x; K) &- \frac{1}{\pi} \sum_{m=0}^\infty K^m(K \log K) \int_{-1}^{+1} f(t)H_m(x, 0, t) \, dt \\ &+ \frac{1}{\pi} \sum_{m=0}^\infty K^{m+1} \int_{-1}^{+1} f(t)G_m(x, 0, t) \, dt = \sum_{m=0}^\infty g_m(x)K^m \quad \text{on} \quad \mid x \mid < 1. \end{aligned} \tag{3.11}$$

Now we remark that the power products $K^m(K \log)^n$ can be ordered according to increasing degree of vanishing as $K \to 0$, i.e.

$$\lim_{K\to 0} \frac{K^{m'}(K \log K)^{n'}}{K^m(K \log K)^n} = 0 \text{ if } m' + n' > m + n \text{ or } m' + n' = m + n \text{ and } n > n'.$$

We proceed to develop the solution of (3.11) in the set of these power products. Passing to limit $K = 0$ in (3.11) we find,

$$f_{00}(x) = \lim_{K\to 0} f(x; K) = g_0(x).$$

Then (3.11) becomes,

$$f(x; K) - f_{00}(x) - \frac{1}{\pi} K \log K \int_{-1}^{+1} f(t; k) H_0(x, 0, t)\, dt = o(K \log K) \quad \text{on} \quad |x| < 1.$$

Dividing by $K \log K$ and passing to limit $K = 0$,

$$f_{01}(x) = \lim_{K \to 0} \left[\frac{f(x; K) - f_{00}(x)}{K \log K} \right] = \frac{1}{\pi} \int_{-1}^{+1} f_{00}(t)\, dt = \frac{1}{\pi} \int_{-1}^{+1} g_0(t)\, dt.$$

Reentering in (3.11),

$$f(x; K) - f_{00}(x) - f_{01}(x) K \log K + \frac{K}{\pi} \int_{-1}^{+1} f(t; k) G_0(x, 0, t)$$
$$= g_1(x) K + o(K) \quad \text{on} \quad |x| < 1.$$

Dividing by K and passing to limit $K = 0$,

$$f_{10}(x) = \lim_{K \to 0} [f(x; K) - f_{00}(x) - f_{01}(x) K \log K]/K = g_1(x) - \frac{1}{\pi} \int_{-1}^{+1} f_{00}(t) G_0(x, 0, t)\, dt.$$

We have indicated here the first three steps of a process which can be continued indefinitely. The process leads ultimately to,

$$f(x; K) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn}(x) K^m (K \log K)^n. \tag{3.12}$$

That the successive $f_{mn}(x)$ can be determined by recursion one sees as follows. We say $f(x; K)$ has an estimate of degree (m, n) if,

$$f(x; K) = P(K, K \log K) + o(K^m (K \log K)^n),$$

where P is a polynomial (with coefficients depending on x) of degree (m, n). Since,

$$KG = -K \log K + o(K \log K)$$

we see that the product of KG with a polynomial of degree (m, n) is a polynomial of degree $(m, n + 1)$ plus terms $o(K^m (K \log K)^{n+1})$. Suppose then that we have shown $f(x; K)$ to have an estimate of degree (m, n), that is have computed $f_{00}, f_{01}, \cdots, f_{mn}$ in (3.12). Substituting this estimate in the integral in (3.11) we obtain a polynomial of degree $(m, n + 1)$ plus terms of $o([K^m (K \log K)^{n+1}]$, with coefficients determined by the known quantities, $f_{00}, \cdots, f_{mn}$. Right hand side of (3.11) has estimates of degree $(m, 0)$ for all m, hence (3.11) yields for $f(x; K)$ an estimate of degree $(m, n + 1)$.

Substituting the series (3.12) and (3.10) in the integral representation (3.2) it is readily seen that $u(x, y; K)$ has the form,

$$Ku(x, y; K) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_m(x, y) K^m (K \log K)^n. \tag{3.13}$$

4. Convergence of the asymptotic developments.* The development (3.13) is asymptotic and the question arises as to whether the series actually converges to yield the function $u(x, y; K)$. For the special case $g(x) = -e^{iKx}$, which governs the diffraction of waves by a rigid dock, an alternative procedure exists for obtaining the development (3.13). This second technique answers the convergence question while at the same time pointing up the relations which exist between the coefficients of the development.

*The idea of this section was suggested to the author by Professor Hans Lewy.

For simplicity we consider only the even part of the solution that is the solution of Problem (II) with $g(x) = -\cos Kx$. The odd part could be treated in the same manner. The solution has the representation (3.2) with $f(x; K)$ a solution of the equation,

$$f(x; K) + \frac{K}{\pi} \int_{-1}^{+1} f(t; K) G(x, 0, t; K)\, dt = -\cos Kx \quad \text{on} \quad x < 1. \tag{4.1}$$

$G(x, 0, t; K)$ is defined as a function of complex K by (3.10), that is,

$$G(x, 0, t; K) = -\{\cos K \mid x - t \mid\} \log K + \sum_{m=0}^{\infty} G_m(x, 0; t) K^m. \tag{4.2}$$

The usual method of successive approximations applied to (4.1) shows that the solution, as a uniform limit of functions analytic in K, is analytic for $\mid K \mid \neq 0$ sufficiently small. Let K^+, K^- denote the image points of K on a logarithmic Riemann surface of K after positive and negative circuits respectively of the origin. By (4.2),

$$G(x, 0, t; K^+) - G(x, 0, t; K) = -2\pi i \cos K \mid x - t \mid$$

hence if we form (4.1) for K^+ we find,

$$\begin{aligned} f(x, K^+) + \frac{K}{\pi} \int_{-1}^{+1} f(t, K^+) G(x, 0, t; K)\, dt \\ - 2iK \int_{-1}^{+1} f(t, K^+) \cos K \mid x - t \mid dt = -\cos Kx. \end{aligned} \tag{4.3}$$

Subtracting (4.1) from (4.3) and noting that f is even with respect to t

$$\begin{aligned} f(x, K^+) - f(x, K) + \frac{K}{\pi} \int_{-1}^{+1} [f(t, K^+) - f(t, K)] G(x, 0, t; K)\, dt \\ = 2iK \int_{-1}^{+1} f(t, K^+) \cos K \mid x - t \mid dt = 2iK \cos Kx \int_{-1}^{+1} f(t, K^+) \cos Kt\, dt \end{aligned}$$

hence the difference, $f(x, K^+) - f(x, K)$ satisfies, except for a multiplicative constant the same integral equation (4.1), as $f(x, K)$ itself. Therefore,

$$f(x, K^+) - f(x, K) = 2iK f(x, K) \int_{-1}^{+1} f(t, K^+) \cos Kt\, dt$$

or,

$$f(x, K) - f(x, K^-) = 2iK f(x; K^-) \int_{-1}^{+1} f(t, K) \cos Kt\, dt. \tag{4.4}$$

The function $f(x, K)$ thus satisfies a non-linear difference equation in K. If we let,

$$\lambda(K) = \int_{-1}^{+1} f(t, K) \cos Kt\, dt$$

we find by multiplying (4.4) by $\cos Kx$ and integrating,

$$\lambda(K) - \lambda(K^-) = 2iK \lambda(K) \lambda(K^-). \tag{4.5}$$

This difference equation for $\lambda(K)$ is still non-linear but we can form from it a linear equation. As in the derivation of (3.12) we have from (4.1),

$$f(t, K) = -1 + o(1) \quad \text{as} \quad K \to 0.$$

Hence,

$$\lambda(K) = -2 + o(1) \quad \text{as} \quad K \to 0 \tag{4.6}$$

so that $\lambda(K)$ [and also $\lambda(K^-)$] is different from 0 for K sufficiently small. Thus for small K, (4.5) can be divided by $\lambda(K)\, \lambda(K^-)$ to yield,

$$\frac{1}{\lambda(K^-)} - \frac{1}{\lambda(K)} = 2iK.$$

It follows that,

$$\tau \equiv \frac{1}{\lambda(K)} = -\frac{K}{\pi} \log K + S(K) \quad \text{or} \quad \lambda(K) = \frac{1}{S(K) - \dfrac{K}{\pi} \log K}, \tag{4.7}$$

where $S(K)$ is single-valued, and being bounded by (4.6), is a power series, i.e.,

$$S(K) = \frac{1}{2}\left(1 + \sum_{n=1}^{\infty} S_n K^n\right). \tag{4.8}$$

Substituting (4.7) in (4.4) we find,

$$f(x, K) - \left[1 + \frac{2iK}{S(K) - \dfrac{K}{\pi} \log K}\right] f(x, K^-) = 0. \tag{4.9}$$

Now observe that,

$$\tau(K^-) = S(K) - \frac{K}{\pi} \log K + 2iK = \tau(K)\left[1 + \frac{2iK}{S(K) - \dfrac{K}{\pi} \log K}\right].$$

Also $\tau(0) \neq 0$, hence for sufficiently small K (4.9) can be written

$$\tau(K) f(x, K) - \tau(K^-) f(x, K^-) = 0$$

i.e., $\tau(K) f(x, K)$ is a single-valued function, $p(x; K)$, which is continuous at $K = 0$, and thus,

$$f(x, K) = \frac{p(x, K)}{S(K) - \dfrac{K}{\pi} \log K} \qquad p(x, K) = \sum_{n=0}^{\infty} p_n(x) K^n. \tag{4.10}$$

From (4.10) two interesting facts may be observed. First we can write, since $S(0) = \frac{1}{2}$,

$$f(x, K) = 2p(x; K) \frac{1}{1 - Z(K)},$$

where

$$Z(K) = 2 \frac{K}{\pi} \log K - \sum_{n=1}^{\infty} S_n K^n.$$

Since $| Z(K) | < 1$ for K sufficiently small, say $| K | < K_0$, we have,

$$f(x, K) = 2p(x; K) \sum_{n=0}^{\infty} (Z(K))^n \tag{4.11}$$

which it is readily seen leads to a *convergent* double power series in K and $(K \log K)$.

Second we see by (4.10) and (4.11) that $f(x, K)$ is completely determined by the two analytic functions $p(x, K)$ and $S(K)$. That is to say in the double series development

$$f(x, K) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}(x) K^m (K \log K)^n \tag{4.12}$$

for $f(x, K)$ there are at most two simple infinities of the coefficients a_{mn} which are sufficient to determine the others. In reality the analytic functions $p(x; K)$ and $S(K)$ are related so that only a single infinity of the coefficients need be determined. One sees this by using (4.11) to express the S_n and $p_n(x)$ as functionals of $f(x: K)$, i.e., of the $a_{ij}(x)$. If the operation in (4.11) is carried out and the series (4.12) is substituted on the left, we obtain a series of relations, the first few of which we write down here:

$$2p_0(x) = a_{00}(x),$$

$$\frac{4p_0(x)}{\pi} = a_{01}(x),$$

$$2p_1(x) - 2p_0(x)S_1 = a_{10}(x),$$

$$\frac{8}{\pi^2} p_0(x) = a_{02}\ ,$$

$$\frac{4p_1(x)}{\pi} - \frac{8S_1p_0(x)}{\pi} = a_{11}(x).$$

We see from these relations first that the $p_n(x)$ are determined by the $a_{i0}(x)$ and the S_n and then that the $a_{ij}(x)$ for $j \geq 1$ are determined by the $a_{i0}(x)$. The $p_n(x)$ and $\{S_n\}$ and hence $f(x, K)$ are completely specified if one knows the values of the $a_{i0}(x)$ $i = 0, 1, 2, \cdots$.

The intermediate step of computing the $a_{ij}(x)$ may be eliminated as follows. Substituting (4.10) and (3.10) in (4.1) yields,

$$p(x, K) + \frac{1}{\pi} \int_{-1}^{+1} p(t, K)\{-K \log K (\cos K \mid x - t \mid) + \sum_{m=0}^{\infty} G_m(x, 0, t) K^{m+1}\}\, dt$$
$$= -\cos Kx \left[S(K) - \frac{K}{\pi} \log K \right] \quad \text{on} \quad |x| < 1.$$

We can equate here terms which involve $\log K$ and those which do not, obtaining

$$-\frac{1}{\pi} \int_{-1}^{+1} p(t, K) \cos K \mid x - t \mid dt = \frac{1}{\pi} \cos Kx \quad \text{on} \quad |x| < 1 \tag{4.13}$$

$$p(x, K) + \frac{1}{\pi} \int_{-1}^{+1} p(t, K) \sum_{m=0}^{\infty} G_m(x, 0, t) K^{m+1}\, dt = -S(K) \cos Kx. \tag{4.14}$$

Since $p(x, K)$ is an even function of x (4.13) may be rewritten,

$$\int_{-1}^{+1} p(t, K) \cos Kt\, dt = -1. \tag{4.15}$$

Set $p^*(x, K) = p(x, K)/S(K)$. Recalling that $S(0) \neq 0$ $p^*(x, K)$ is an analytic function of K for K small, i.e.,

$$p^*(x, K) = \sum_{n=0}^{\infty} p_n^*(x) K^n.$$

Substituting in (4.14) and equating coefficients of K^r, $r = 0, 1, 2, \cdots$, yields, $p_0^* = -1$,

$$\begin{aligned} p_r^*(x) + \frac{1}{\pi} \sum_{n=0}^{r-1} \int_{-1}^{+1} p_n(t) G_{r-1-n}(x, 0, t)\, dt & \\ = \frac{(-1)^{r+1}}{2r!} x^{2r} \quad \text{if } r \text{ is even}, \qquad r > 0 & \qquad (4.16) \\ = 0 \qquad\qquad \text{if } r \text{ is odd}. & \end{aligned}$$

Equation (4.16) determines the $p_r(x)$ recursively. For example:

$$p_0^*(x) = -1,$$

$$p_1^*(x) = -\frac{1}{\pi} \int_{-1}^{+1} p_0^*(t) G_0(x, 0, t)\, dt = \frac{2}{\pi} (\pi i - \gamma),$$

$$-\frac{1}{\pi} [(1 + x) \log (1 + x) + (1 - x) \log (1 - x) - 2x].$$

Once the $p_r^*(x)$ are found Eq. (4.15) can be used to determined $S(K)$. Substituting $p(x, K) = S(K) p^*(x; K)$ in (4.15) and equating coefficients of K^i yields,

$$\begin{aligned} \frac{1}{2} \sum_{m=0}^{[i/2]} \frac{(-1)^m}{2m!} \int_{-1}^{+1} p_{i-2m}^*(t) t^{2m}\, dt\, * & \\ + \sum_{n=1}^{i} \sum_{m=0}^{[i-n/2]} S_n \frac{(-1)^m}{2m!} \int_{-1}^{+1} p_{i-n-2m}^* t^{2m}\, dt = 0, \qquad i = 1, 2, 3, \cdots. & \qquad (4.17) \end{aligned}$$

Equation (4.17) determines the S_n recursively since the coefficient of S_i is,

$$\int_{-1}^{+1} p_0^*(t)\, dt = -2 \neq 0.$$

For example:

$$S_1 = \frac{1}{4} \int_{-1}^{+1} p_1^*(t)\, dt = i - \frac{\gamma}{\pi} - \frac{1}{2} (\log 2 - 1).$$

Once $S(K)$ and $p^*(x; K)$ are found, $p(x; K)$ is obtained as their product.

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* $[z]$ denotes greatest integer not greater than z.

BOOK REVIEWS

(*Continued from p. 146*)

of statistics. Canonical analysis is discussed and illustrated by examples in Chapter 5, while Chapters 6 and 8 deal with distribution problems and tests of multivariate hypotheses. Among other topics treated in this book are tests of homogeneity, discriminant functions and some history of multivariate analysis. An extensive bibliography and a set of exercises end the book.

ULF GRENANDER

The analysis of multiple time-series. By M. H. Quenouille. Hafner Publishing Co., New York, 1957. 105 pp. $4.75.

The last decades have witnessed rapid progress in statistical time series analysis, as far as a single series is concerned. This was made possible by the results in probability theory describing the probability structure of time series of various types. For multiple (vector valued) time series the progress has been slower, one of the reasons perhaps being that we know less about the basic properties of vector valued stochastic processes. It is typical that the linear prediction problem solved in the early forties for scalar processes, is still not quite settled in the vector case, although recent work of Wiener and Masani may change this.

In this book the author collects available methods and develops new ones for analyzing multiple time series. His starting point is the assumption that the underlying process is of the finite parameter type: a moving average, an autoregressive scheme or a combination of both. This restriction may be reasonable in the present preliminary stage of development of the theory.

After a brief discussion of these schemes and of their covariance properties and identification problems, the author studies the numerical behavior of five artificially generated series. The next chapter, Practical complications, contains a discussion of how various factors, may influence the anaylsis. Such situations occur e.g. when 1) the errors are serially correlated 2) the sample size is too small, so that large sample approximations are inadequate or 3) there are unknown trends or seasonal variation. After two chapters on estimation and testing the author investigates an econometric example in detail trying to apply the methods described in the book.

The emphasis of the book is on the formal development of statistical techniques, and one would have appreciated a more complete critical examination of the validity of the methods used. However it is clearly too early to ask for definitive and convincing general methods of analysis of multiple time series, and the need for further work is obvious. At present we will have to use the tools of analysis available to us, even if they leave a good deal to be desired. When applied with some caution they should enable us to extract some information from the data.

This book is the first in the new series "Griffin's statistical monographs and courses" in which has also appeared the book on multivariate analysis reviewed elsewhere in this journal.

ULF GRENANDER

Applied mathematics for engineers and physicists. By Louis A. Pipes. McGraw-Hill Book Co., New York, Toronto, London, 1958. xi + 723 pp. $8.75.

This second edition of a book first published in 1946 remains essentially the same as the first edition as regards subject matter, style and purpose. New material has been added to most chapters making the book somewhat larger than its predecessor. Particularly there is a greater emphasis on methods of numerical analysis some of which have been developed during the last decade as a result of the stimulus furnished by the high speed computers.

The sections on theory and applications of matrices and of Laplace transforms have been revised and contain much material not usually found in a text of this type. The book continues, however, to give recipes for solving specific problems with considerable disdain for sound mathematics.

G. F. NEWELL

(*Continued on p. 172*)

MATRIX DIFFERENTIAL SYSTEMS WITH A PARAMETER IN THE BOUNDARY CONDITIONS AND RELATED VIBRATION PROBLEMS*

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In a series of papers by Moshinsky [1, 3] and Adem and Moshinsky [2] the description of some physical processes has been given by a column matrix or vector instead of a single function, achieving a simplification in the mathematical solution of such processes and making it possible to solve a variety of problems. In the examples considered in [2, 3], for example, the above formulation leads to a matrix eigenvalue problem which is a generalization of the Sturm-Liouville problem and that can be treated along similar lines.

In this paper, by describing a vibratory system in the aforementioned way, we are led to a more general eigenvalue problem that contains a parameter both in the differential equations and in the boundary conditions. This problem can be considered as a generalization of the corresponding scalar case, examples of which have been considered recently by Bauer [4] and Morgan [5].

In the first part of this paper, we give examples of vibratory systems that give rise to this new type of eigenvalue problem.

In the second part of this paper, we formulate and discuss the general matrix differential system and obtain some properties of the eigenvalues and eigen column matrices.

I. Vibrating string with a concentrated mass at a point. Consider a string with a concentrated mass at a point which separates two regions of different density and length of which the string is composed. The string is fixed at the ends. We wish to find the transverse displacements of the string if it is displaced initially into a position and released from rest at this position with no external forces acting.

Let the length of the string be $L_1 + L_2$, and M the concentrated mass at a distance L_1 from the left end point of the string. We shall designate I and II as the regions to the left and right of the concentrated mass respectively, and Y_1 and Y_2 as the corresponding displacements. Region I has a density ρ_1, and a length L_1, while II has a density ρ_2 and a length L_2. The space-independent variable shall be called x_1 in I and x_2 in II; x_1 increases from left to right, while x_2 increases from right to left, so that at the end points we have $x_1 = 0$, $x_2 = 0$ and at the point with the concentrated mass, $x_1 = L_1$, $x_2 = L_2$.

Considering that in both regions the equation of a vibrating string applies and considering boundary and initial conditions we obtain the following differential equations and boundary conditions:

*Received June 11, 1958. Numbers in square brackets refer to the bibliography at the end of the paper.

$$\frac{\partial^2 Y_i}{\partial x_i^2} = \frac{1}{C_i^2}\frac{\partial^2 Y_i}{\partial t^2} \qquad (i = 1, 2), \tag{1}$$

$$\left.\begin{aligned} Y_1(0, t) &= Y_2(0, t) = 0 \\ Y_1(L_1, t) &= Y_2(L_2, t) \\ \left(\frac{\partial Y_1}{\partial x_1}\right)_{x_1=L_1} + \left(\frac{\partial Y_2}{\partial x_2}\right)_{x_2=L_2} &= -\frac{M}{T}\left(\frac{\partial^2 Y_1}{\partial t^2}\right)_{x_1=L_1} \end{aligned}\right\}, \tag{2}$$

$$Y_1(x_1, 0) = F_1(x_1); \qquad Y_2(x_2, 0) = F_2(x_2), \tag{3}$$

$$\left(\frac{\partial Y_1}{\partial t}\right)_{t=0} = \left(\frac{\partial Y_2}{\partial t}\right)_{t=0} = 0, \tag{4}$$

where $C_i^2 = T/\rho_i$ $(i = 1, 2)$; T is the tension on the string; $F_1(x_1)$ and $F_2(x_2)$ are the initial displacements prescribed at regions I and II respectively.

Applying the change of independent variables

$$x_i = L_i x_i' \qquad (i = 1, 2),$$

we have that now the range of x_1' is the same as that of x_2' and we can therefore put $x_1' = x_2' = x$.

Problem (1), (2), (3), (4), becomes:

$$\frac{1}{L_i^2}\frac{\partial^2 y_i}{\partial x^2} = \frac{1}{C_i^2}\frac{\partial^2 y_i}{\partial t^2} \quad \text{for} \quad 0 < x < 1, \tag{1'}$$

$$\left.\begin{aligned} y_1(0, t) &= y_2(0, t) = 0 \\ y_1(1, t) &= y_2(1, t) \\ \frac{1}{L_1}\left(\frac{\partial y_1}{\partial x}\right)_{x=1} + \frac{1}{L_2}\left(\frac{\partial y_2}{\partial x}\right)_{x=1} &= -\frac{M}{T}\left(\frac{\partial^2 y_1}{\partial t^2}\right)_{x=1} \end{aligned}\right\}, \tag{2'}$$

$$y_1(x, 0) = f_1(x); \qquad y_2(x, 0) = f_2(x), \tag{3'}$$

$$(\partial y_1/\partial t)_{t=0} = (\partial y_2/\partial t)_{t=0}, \tag{4'}$$

where

$$x = x_i/L_i, \qquad y_i(x, t) = Y_i(x_i, t)$$

and

$$f_i(x) = F_i(x_i).$$

Let

$$y_i(x, t) = y_i(x)\cos\omega t$$

then problem (1′), (2′) becomes:

$$\frac{1}{L_i^2}\frac{d^2 y_i}{dx^2} + \frac{\omega^2}{C_i^2}y_i = 0 \qquad (i = 1, 2), \tag{5}$$

$$y_1(0) = y_2(0) = 0, \tag{6}$$

$$\left.\begin{aligned} y_1(1) &= y_2(1), \\ \left(\frac{1}{L_1}\frac{dy_1}{dx} + \frac{1}{L_2}\frac{dy_2}{dx}\right)_{x=1} - \frac{M}{T}\omega^2 y_1(1) &= 0 \end{aligned}\right\}. \tag{7}$$

The solution of Eq. (5) is

$$y_i(x) = A_i \cos(\omega L_i x/C_i) + B_i \sin(\omega L_i x/C_i),$$

where A_i, B_i and ω are going to be determined from the boundary conditions.

From conditions (6) it follows that $A_i = 0$.

From conditions (7) we get the frequency equation:

$$\left.\begin{aligned}(1/C_2)\sin(\omega L_1/C_1)\cos(\omega L_2/C_2) + (1/C_1)\cos(\omega L_1/C_1)\sin(\omega L_2/C_2)&\\ - (M\omega/T)\sin(\omega L_1/C_1)\sin(\omega L_2/C_2) = 0&\end{aligned}\right\} \tag{8}$$

whose roots are real and form an infinite denumerable set. Let the positive roots of Eq. (8) be $\omega_1, \omega_2, \cdots, \omega_n, \cdots$. They are the eigenvalues of the problem (5), (6), (7).

To each eigenvalue there corresponds a solution:

$$\left.\begin{aligned} y_{1j}(x) &= \sin(\omega_j L_1 x/C_1)\\ y_{2j}(x) &= \frac{\sin(\omega_j L_1/C_1)}{\sin(\omega_j L_2/C_2)}\sin(\omega_j L_2 x/C_2)\end{aligned}\right\}, \tag{9}$$

where we have used the notation $y_{1j}(x)$, $y_{2j}(x)$ instead of $y_1(x)$, $y_2(x)$ to distinguish that it is a solution corresponding to the ω_j eigenvalue. $y_{1j}(x)$ and $y_{2j}(x)$ can be considered as the components of a column matrix (vector) which will be denoted by $\mathbf{y}_j(x)$ and called eigen column matrix:

$$\mathbf{y}_j(x) = \begin{bmatrix} y_{1j}(x)\\ y_{2j}(x)\end{bmatrix}.$$

Proceeding as usual, we assume as solution of our problem an infinite series of the type:

$$\mathbf{y}(x, t) = \sum_{j=1}^{\infty} B_j \mathbf{y}_j(x)\cos\omega_j t.$$

This series is a solution of (1′), (2′), (4′) provided it is convergent and admits two successive term by term differentiations with respect to t and x. Assuming that this condition is satisfied, there remains, then, to determine the constants B_j, so that (3′) is satisfied:

$$\mathbf{f}(x) \equiv \begin{bmatrix} f_1(x)\\ f_2(x)\end{bmatrix} = \sum_{j=1}^{\infty} B_j \begin{bmatrix} y_{1j}(x)\\ y_{2j}(x)\end{bmatrix} \equiv \sum_{j=1}^{\infty} B_j \mathbf{y}_j(x). \tag{10}$$

Assuming the validity of expansion (10) the coefficients B_j can easily be determined and for it we shall establish first the orthogonality relation of the eigen column matrices corresponding to the problem (5), (6), (7).

Since $\mathbf{y}_k(x)$ and $\mathbf{y}_p(x)$ are solutions of (5) corresponding to ω_k and ω_p respectively, the following relation is satisfied:

$$\sum_{i=1}^{2}\frac{1}{L_i}\left(y_{ik}\frac{d^2y_{ip}}{dx^2} - \frac{d^2y_{ik}}{dx^2}y_{ip}\right) = \sum_{i=1}^{2}\frac{L_i}{C_i^2}(\omega_k^2 - \omega_p^2)y_{ik}y_{ip}\,.$$

Therefore

$$\sum_{i=1}^{2}\frac{1}{L_i}\left[y_{ik}\frac{dy_{ip}}{dx} - y_{ip}\frac{dy_{ik}}{dx}\right]_0^1 = (\omega_k^2 - \omega_p^2)\int_0^1 \sum_{i=1}^{2}\frac{L_i}{C_i^2}y_{ik}y_{ip}\,dx.$$

From boundary conditions (6) and (7) it follows that

$$\sum_{i=1}^{2} \int_0^1 \frac{L_i}{C_i^2} y_{ik}(x) y_{ip}(x)\, dx + \frac{M}{T} y_{1k}(1) y_{1p}(1) = 0 \tag{11}$$

or with matrix notation:

$$\int_0^1 \mathbf{y}_k'(x) \mathbf{H} \mathbf{y}_p(x)\, dx + \mathbf{y}_k'(1) \mathbf{J} \mathbf{y}_p(1) = 0$$

where

$$\mathbf{H} = \begin{bmatrix} L_1/C_1^2 & 0 \\ 0 & L_2/C_2^2 \end{bmatrix}; \qquad \mathbf{J} = \begin{bmatrix} M/T & 0 \\ 0 & 0 \end{bmatrix}$$

and $\mathbf{y}_k'(x)$ denotes the transposed of $\mathbf{y}_k(x)$.
This is an orthogonality condition for the eigen column matrices of problem (5), (6), (7).

Assuming that the series (10) can be integrated term by term after being multiplied by $\mathbf{y}_i'(x)\mathbf{H}$, we can now, using (11), determine the coefficients B_i in (10), which are given by

$$B_i = \frac{\displaystyle\int_0^1 [(L_1/C_1^2) f_1(x) y_{1i}(x) + (L_2/C_2^2) f_2(x) y_{2i}(x)]\, dx + (M/T) f_1(1) y_{1i}(1)}{\displaystyle\int_0^1 [(L_1/C_1^2) y_{1i}^2(x) + (L_2/C_2^2) y_{2i}^2(x)]\, dx + (M/T) y_{1i}^2(x)} \tag{12}$$

or more briefly with matrix notation

$$B_i = \frac{\displaystyle\int_0^1 \mathbf{y}_i'(x) \mathbf{H} \mathbf{f}(x)\, dx + \mathbf{f}'(1) \mathbf{J} \mathbf{y}_i(1)}{\displaystyle\int_0^1 \mathbf{y}_i'(x) \mathbf{H} \mathbf{y}_i(x)\, dx + \mathbf{y}_i' \mathbf{J} \mathbf{y}_i(1)}.$$

Substituting (9) into (12) we obtain

$$\begin{aligned} \frac{1}{2}\left[\frac{L_1}{C_1^2} + \sin^2 \frac{\omega_i L_1}{C_1}\right]&\left[\frac{L_2}{C_2^2 \sin^2 (\omega_i L_2/C_2)} + \frac{M}{T}\right] B_i \\ &= \frac{L_1}{C_1^2} \int_0^1 f_1(x) \sin \frac{\omega_i L_1 x}{C_1}\, dx + \frac{L_2 \sin (\omega_i L_1/C_1)}{C_2^2 \sin (\omega_i L_2/C_2)} \int_0^1 f_2(x) \sin \frac{\omega_i L_2 x}{C_2}\, dx \\ &\qquad + \frac{M}{T} f_1(1) \sin \frac{\omega_i L_1}{C_1}, \end{aligned} \tag{13}$$

and the formal solution to the problem (1), (2), (3), (4), is:

$$\begin{bmatrix} Y_1(x_1, t) \\ Y_2(x_2, t) \end{bmatrix} = \sum_{i=1}^{\infty} B_i \begin{bmatrix} \sin (\omega_i x_1/C_1) \\ \dfrac{\sin (\omega_i L_1/C_1)}{\sin (\omega_i L_2/C_2)} \sin (\omega_i x_2/C_2) \end{bmatrix} \cos \omega_i t,$$

where ω_i are the positive roots of Eq. (8) and the B_i's are given by (13).

II. Solution of other vibrating systems. The above method can be applied to the case of a string with n concentrated masses $M_1, M_2, \cdots, M_n$ which separate $n + 1$

regions of different density $\rho_1, \rho_2, \cdots, \rho_n$; and different length $L_1, L_2, \cdots, L_n$. In this case we describe the system by $n + 1$ functions $Y_1, Y_2, \cdots$, which by a change of variable is always possible to have in the same domain, for example, $0 \leq x \leq 1, t \geq 0$. We call the $n + 1$ regions from left to right, I, II, III, $\cdots$ and, for example, in the regions labeled by an odd number let the space variable increase from left to right, while in those labeled by an even number let it increase from right to left. We are led to a system of $n + 1$ differential equations of the type (1′) with boundary conditions similar to conditions (2′), (3′), (4′). In fact, we are led to a matrix differential system which is a particular case of the one discussed in the next section

Since the solution of the mathematical problem corresponding to a vibrating string also solves analogous problems of torsional vibrations of shafts of circular section and longitudinal vibrations of bars, the above method yields exact solutions (without neglecting the mass of the bars, as is usual in Theory of Vibrations) of such systems when they have several concentrated masses, which separate bars of different elasticity, section and length

III. General matrix differential system. The problem (5), (6), (7) as well as those that arise in the examples of Sec. II, are particular cases of the system formed by the general self-adjoint matrix differential equation [6]

$$L(\mathbf{y}) \equiv \frac{d}{dx}\left(\mathbf{P}\frac{d\mathbf{y}}{dx}\right) + \mathbf{Q}\frac{d\mathbf{y}}{dx} + \left(\frac{1}{2}\frac{d\mathbf{Q}}{dx} + \mathbf{R}\right)\mathbf{y} = \lambda\mathbf{W}\mathbf{y} \tag{14}$$

and boundary conditions of the type

$$\left.\begin{aligned} (\mathbf{A} + \lambda\mathbf{B})\mathbf{y}(0) + \mathbf{C}(d\mathbf{y}/dx)_{x=0} &= 0 \\ (\mathbf{D} + \lambda\mathbf{E})\mathbf{y}(1) + \mathbf{F}(d\mathbf{y}/dx)_{x=1} &= 0 \end{aligned}\right\}, \tag{15}$$

where $\mathbf{P}$, $\mathbf{R}$ and $\mathbf{W}$ are $n \times n$ symmetric matrices, $\mathbf{Q}$ $n \times n$ antisymmetric matrix; $\mathbf{P}$, $\mathbf{R}$, $\mathbf{W}$, $\mathbf{Q}$ are real functions of x; the unknown $\mathbf{y}$ is a column matrix of n components; λ is a scalar parameter; $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, are $n \times n$ constant real matrices. We assume that $\mathbf{P}$, $\mathbf{R}$, $\mathbf{W}$, $\mathbf{Q}$, $d\mathbf{P}/dx$, $d\mathbf{Q}/dx$ are continuous and that the determinant of $\mathbf{P}$ is different from zero in $0 < x < 1$.

From (15) we obtain

$$\begin{aligned} &\int_0^1 [\mathbf{y}_i' L(\mathbf{y}_j) - L'(\mathbf{y}_i)\mathbf{y}_j]\, dx \\ &\qquad \equiv [\mathbf{y}_i'\mathbf{P}(d\mathbf{y}_j/dx) - (d\mathbf{y}_i'/dx)\mathbf{P}\mathbf{y}_j + \mathbf{y}_i'\mathbf{Q}\mathbf{y}_j]_0^1 = (\lambda_j - \lambda_i)\int_0^1 \mathbf{y}_i'\mathbf{W}\mathbf{y}_j\, dx, \end{aligned} \tag{16}$$

where $\mathbf{y}_i$ and $\mathbf{y}_j$ are two different solutions (eigen column matrices) corresponding to the eigenvalues λ_i and λ_j respectively; and y_i' and $L'(\mathbf{y}_i)$ denote the transposed of $\mathbf{y}_i$ and $L(\mathbf{y}_i)$ respectively (i.e., they are row matrices of n elements).

The problem that we are going to deal with is such that by substituting the boundary conditions (15) into (16) we obtain a relation of the type:

$$[\mathbf{y}_i, \mathbf{y}_j] \equiv \int_0^1 \mathbf{y}_i'(x)\mathbf{W}(x)\mathbf{y}_j(x)\, dx + \mathbf{y}_i'(0)\mathbf{M}\mathbf{y}_j(0) + \mathbf{y}_i'(1)\mathbf{N}\mathbf{y}_j(1) = 0, \tag{17}$$

where $\mathbf{M}$ and $\mathbf{N}$ are constant $n \times n$ matrices. Relation (17) can be considered as an orthogonality relation that the eigen column matrices of the problem (14), (15) satisfy.

We are more familiar with orthogonality relations in which the last two terms of the left member of Eq. (17) are zero. Such is the case in the self-adjoint problem of the type (14), (15) but in which the parameter λ does not appear in the boundary conditions (i.e., $B = E = 0$). This problem has been extensively studied by G. D. Birkhoff and R. E. Langer [7]; and also by Moshinsky and the author in connection with self-adjointness [6], and examples of it have been given in [2, 3]. As far as the present writer is aware the problem (14), (15) has only been studied in the scalar case (i.e., when $n = 1$) and some papers by Bauer and Morgan have appeared recently dealing with it [4, 5].

Now that we have formulated the mathematical problem (14), (15) the next step would be to develop a theory dealing with properties of eigenvalues and eigen column matrices and with the possibility of expanding an arbitrary column matrix in terms of the eigen column matrices, as is needed for the applications, as, for example, in the problems mentioned in the above sections.

Proceeding as W. F. Bauer does in the scalar case [4], we shall give the following definitions:

D.1. A column matrix $\mathbf{u}(x)$ shall be called V-column matrix if it is real, not identically zero, of class C^1, and satisfies the boundary conditions (15).

D.2. The inner product corresponding to the eigenvalue problem (14), (15) is defined as

$$[\mathbf{h}, \mathbf{g}] \equiv \int_0^1 \mathbf{h}'(x)\mathbf{W}(x)\mathbf{g}(x)\,dx + \mathbf{h}'(0)\mathbf{M}\mathbf{g}(0) + \mathbf{h}'(1)\mathbf{N}\mathbf{g}(1), \tag{18}$$

where $\mathbf{h}(x)$ and $\mathbf{g}(x)$ are column matrices whose components are bounded integrable functions.

D.3. The eigenvalue problem (14), (15) is normal if for every V-column matrix $\mathbf{u}(x)$ we have

$$[\mathbf{u}, \mathbf{u}] > 0.$$

The following results can be established:

T.1. A sufficient condition for the problem (14), (15) to be normal is that $\mathbf{W}$ be a non-zero diagonal matrix whose components are non-negative in the fundamental interval and $\mathbf{N}$ and $\mathbf{M}$ diagonal matrices whose components are non-negative.

T.2. If $\mathbf{W}$, $\mathbf{N}$ and $\mathbf{M}$ satisfy the same condition as in T.1., the eigenvalue problem (14), (15) has only real eigenvalues. The proof of T.2., is analogous to that employed in the scalar case [4].

T.3. If we define as Rayleigh's quotient $R(\mathbf{y})$ for any V-column matrix $\mathbf{y}(x)$

$$R(\mathbf{y}) = \frac{[L(\mathbf{y}), \mathbf{y}]}{[\mathbf{W}\mathbf{y}, \mathbf{y}]},$$

where $L(\mathbf{y})$ stands for the left member of (14), from (14) it follows that $R(\mathbf{y}_i) = \lambda_i$, where $\mathbf{y}_i$ is an eigen column matrix corresponding to the eigenvalue λ_i.

T.4. If $\mathbf{W}$, $\mathbf{N}$ and $\mathbf{M}$ satisfy the same conditions as in T.1., and $[L(\mathbf{y}), \mathbf{y}] > 0$ for every V-column matrix then the eigenvalue problem (14), (15) has only positive eigenvalues. This result follows from T.3.

T.5. Corresponding to each real eigenvalue there is one or at the most, a finite number h_i of linearly independent real eigen column matrices $y_i^{(\alpha)}(x)$, $\alpha = 1, 2, \cdots, h_i$. ($h_i \leq 2n$).

T.6. The eigen column matrices corresponding to different eigenvalues are orthogonal with respect to the inner product defined by (18). The eigen column matrices corresponding to the same eigenvalue can be normalized (we assume that the norm of all eigen column matrices is different from zero) and made mutually orthogonal, by the usual procedure [8] so that our set of eigen column matrices satisfies

$$[y_i^{(\alpha)}, y_j^{(\beta)}] = \delta_{\alpha\beta}\, \delta_{ij} \,, \tag{19}$$

where

$$\alpha = 1, 2, \cdots, h_i \,; \qquad \beta = 1, 2, \cdots, h_j \,; \qquad i, j = 1, 2, \cdots$$

and $\delta_{\alpha\beta}$ is equal to 1 for $\alpha = \beta$ and equal to zero for $\alpha \neq \beta$.

T.7. If we assume the validity of an expansion of the type

$$\mathbf{f}(x) = \sum_{n=1}^{\infty} \sum_{\alpha=1}^{h_n} A_n^{(\alpha)} y_n^{(\alpha)}(x), \tag{20}$$

where $\mathbf{f}(x)$ is a column matrix whose components are bounded integrable functions of x [that satisfy certain conditions that ensure the possibility of expansion (20)], and if the series (20) can be integrated term by term after being multiplied by $\mathbf{y}_m'(x)\mathbf{W}(x)$, it is possible, using (19), to obtain the coefficients $A_m^{(\beta)}$, which are given by

$$A_m^{(\beta)} = [\mathbf{f}(x), \mathbf{y}_m^{(\beta)}].$$

The conditions under which the expansion (20) is possible as well as the existence of an infinite denumerable set of eigenvalues, and some properties of eigenvalues and eigen column matrices of the problem (14), (15) will be left unsettled.

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BOOK REVIEWS

(*Continued from p. 164*)

Logical design of digital computers. By Montgomery Phister, Jr. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1958. xvi + 408 pp. $10.50.

This book contains a development of a methodology of functional design of digital circuits and gives examples of application of the techniques to design of digital computer components. Discussion is limited to synchronous circuit design. The book begins with a brief discussion of the system design problem and the logical designers role, and of circuit components and the binary number system. Next, the fundamentals of Boolean Algebra are presented, followed by a description of various methods of simplifying Boolean functions. The general methods of application of Boolean Algebra in sequential circuit design are developed and utilized in a detailed discussion of the design of the major digital computer components; memory unit, input-output unit, arithmetic unit and control unit. Examples of complete computer design are included.

The book provides a compilation of the techniques available in the field of logical design and, because of both content and organization, should be useful to the logical design student and to the practicing designer. The references to literature in the field and the supply of exercises for the student are comprehensive.

The book serves as an excellent compilation of techniques, particularly in its exposition of the methods of reduction of Boolean functions and the general methods of application. However, motivation for the specification of characteristics of the digital computer components is somewhat weak. This fact is implicitly recognized by the author in the discussion of the logical designer's role as one who is responsible for developing the implementation of a set of requirements provided by the system designer: development of characteristics is not fundamentally in the logical design province. The result of the lack of motivation and discussion of the prescription of system parameters limits the usefulness of the book to those who have some experience in the computer field. The book would not be useful, for example, as a text in a first course on digital computer organization and use, nor in a course on computer system design, but should be reserved for use in an advanced course where the objective is solely logical design.

DEAN GILLETTE

Introduction to difference equations. By Samuel Goldberg. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1958. xii + 260 pp. $6.75.

This volume was motivated by a monograph written at the invitation of the Social Science Research Council Committee on the Mathematical Training of Social Scientists. It is intended as an introduction to the theory and application of recurrence relations of the form $u_{n+1} = f(u_n)$ for students and research workers in the social sciences who have only a rudimentary knowledge of mathematics.

It is a pleasure to say that the author has been eminently successful in his task. The volume is not only easily intelligible, but full of important applications which illustrate the significance of this field. The book should certainly be useful as a basis of a course following after one patterned along the lines of the recent book by Kemeny, Snell and Tompson.

Furthermore, the book would be equally useful as an introductory junior or senior course for physical science majors. In these days of narrow specialization, it is essential that research workers in one domain get a glimpse of how scientists in other fields construct their mathematical models.

RICHARD BELLMAN

(*Continued on p. 184*)

ALGEBRAIC TOPOLOGICAL METHODS FOR CONTACT NETWORK ANALYSIS AND SYNTHESIS*

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1. Introduction. The use of Boolean algebra in the analysis and design of contact networks was introduced by C. Shannon [1], and for series-parallel circuits the problem was completely solved. However, for circuits of a more general nature, e.g., bridge networks, the design problem was not solved and the rules given for analysis depended on special geometric constructions carried out on the circuit diagram. The treatment of general circuits was approached from a topological point of view by the author [2] and S. Okada [3]. The method of Okada is analogous to the "loop" or "mesh" method of ordinary circuit analysis while the method of this paper is analogous to the "nodal" method [4] of ordinary circuit theory. It will be shown that contact network theory can be based on topological considerations and topological methods for analysis and synthesis will be given.

A contact network between two terminals is closed for a given state of the contacts if and only if there is a path between the two terminals such that each branch of the path is itself closed. With no loss of generality each branch may be regarded as containing a single contact and a branch is closed or open according to whether the contact is closed or open. The analysis problem consists of finding a formula for the state of the network in terms of the states of the contacts. The synthesis problem is: given such a formula, find a contact network which realizes it, subject usually to some sort of restriction such as minimality of the number of contacts.

The topological preliminaries will be given in Sec. 2. The analysis problem will be treated in Sec. 3 and the conversion of the topological formulas to formulas of Boolean algebra will be carried out in Sec. 4. The design problem will be treated in Sec. 5 and an example of the synthesis method is given in the last section.

2. Topological preliminaries. A graph is defined as a collection of segments or branches denoted by $x_1, x_2, \cdots, x_b$, and nodes which include the endpoints of the branches. The nodes will be denoted by $y_1, y_2, \cdots, y_n$. We consider linear forms in these variables choosing the coefficients from a modulo two arithmetic**, i.e.,

$$\begin{aligned} &0 + 0 = 0\\ &0 + 1 = 1 + 0 = 1\\ &1 + 1 = 0\\ &0 \cdot 0 = 0\\ &0 \cdot 1 = 1 \cdot 0 = 0\\ &1 \cdot 1 = 1 \end{aligned} \tag{1}$$

*Received May 15, 1958.

**The linear forms in the $x_1\ x_2 \cdots x_b$ and the linear forms in $y_1, y_2 \cdots y_n$ each form a group.

The linear forms will be called 0-chains [5, 6], C, if the terms of the sum are nodes, and 1-chains, σ, if the terms are branches. The structure of a graph is known—and the graph can be drawn if the relation of the nodes to the branches is given. This information can be described by the boundary operator ∂ which is defined for segments by

$$\partial x_k = y_r + y_s , \tag{2}$$

if y_r and y_s are the endpoints of x, and for chains by linearity, i.e.,

$$\partial(y_k + y_r) = \partial y_k + \partial y_r . \tag{3}$$

These relations may also be expressed in matrix form by the incidence matrix (4) $A = (A_{rs})$ where

$$A_{rs} = \begin{cases} 1 & \text{if } y_s \text{ is an endpoint of } x_r , \\ 0 & \text{if not} \end{cases} \tag{4}$$

The rows of this matrix correspond to branches and the columns to nodes. There are exactly two non-zero elements in each row in the matrix e.g., for Eq. (2), the kth row will have ones in the rth and sth columns only. If the boundary of a chain is zero the chain will be said to be a cycle. In the bridge circuit of Fig. 1 the 1-chain $\sigma = x_1 + x_2 + x_4$ is a cycle since $\partial\sigma = \partial x_1 + \partial x_2 + \partial x_4 = y_1 + y_2 + y_2 + y_3 + y_3 + y_1 = 0$.

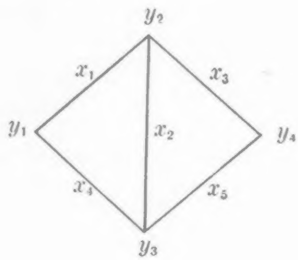

FIG. 1.

The boundary relations may be represented by

$$\partial x_r = \sum_{s=1}^{n} A_{rs} y_s \qquad (r = 1, 2, \cdots , b).$$

The incidence matrix for this bridge is:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{5}$$

An alternative description is also given by the coboundary operator δ defined for nodes by

$$\delta(\text{node}) = \text{sum of the branches having this node as an endpoint,} \tag{6}$$

and for chains by linearity, i.e.,

$$\delta(y_r + y_s) = \delta y_r + \delta y_s . \tag{6'}$$

Thus, in Fig. 1

$$\delta y_2 = x_1 + x_2 + x_3 ,$$

and

$$\delta(y_1 + y_2) = x_1 + x_4 + x_1 + x_2 + x_3 = x_2 + x_4 + x_3 .$$

The incidence matrix A', for this operator is defined by:

$$A'_{rs} = \begin{cases} 1 \text{ if } x_s \text{ has } y_r \text{ as an endpoint} \\ \\ 0 \text{ if not} \end{cases} \tag{7}$$

A comparison with (4) shows that A' is the transpose of A.

In terms of the matrix elements the coboundary operator may be represented by:

$$\delta y_r = \sum_{s=1}^{b} A_{sr} x_s \qquad (r = 1, 2, \cdots, n). \tag{8}$$

A chain is a cocycle if its coboundary is zero. For example, $C = y_1 + y_2 + y_3 + y_4$ is a cocycle. A pair of nodes, y_r and y_s are said to be connected if there is a 1-chain σ such that $\partial\sigma = y_r + y_s$. In Fig. 1, y_1 and y_2 are connected since $\partial x_1 = y_1 + y_2$; and y_1 and y_4 are connected since $\partial(x_1 + x_3) = y_1 + y_4$. One such chain consists of the sum of the segments of any path connecting the nodes in question. Also, two nodes are said to be homologous if they are connected and when a graph is said to be connected it means that every pair of its nodes is connected. The lemma on which our considerations are based is:

Lemma. A zero chain is a cocycle if and only if, for every node in the zero chain, every node connected (homologous) to the given node is also present [6].

For completeness the proof is included. With no loss of generality we may consider only a connected graph. If the 0-chain consists of the sum of all the nodes then since every branch is a term of each of the coboundaries of exactly two nodes it is a consequence of the modulo-two addition that this chain is a cocycle. Conversely, if a node y_k is in a cocycle then every node to which it is connected by a branch must also be in the cocycle; for if not, then the branch of which such a node is an endpoint would be present in the coboundary only once. In general, if y_r is connected to y_k then there is a chain which is a path $x_a + x_b + \cdots + x_k$ such that $\partial(x_a + x_b + \cdots + x_k) = y_k + y_r$, and every node which is an endpoint of a branch of this chain must be in the cocycle since, if this were not the case, some branch would appear only once in the coboundary. Hence, for any given node, the cocycle must contain every node connected to the given node.

3. Topological analysis of contact networks. An immediate consequence of the above lemma is

Theorem I. Two nodes of a graph are connected if and only if there is no cocycle which contains one of these nodes but not the other.

Thus, if we have a graph with n nodes and say we wish to determine whether y_1 is connected with y_n , we form all the 0-chains which contain y_1 but not y_n—there are 2^{n-2}

such chains—and find the coboundaries of these chains. If no coboundary vanishes, i.e., is a cocycle, then the two nodes are connected. We may indicate this symbolically by writing the formal product of these coboundaries with the convention that the product is to be zero if and only if at least one factor is zero, and our result may be stated as: y_1 and y_n are connected if and only if the product of the given coboundaries is not zero.

An explicit formula can be written by listing the integers from zero to $(2^{n-2} - 1)$ in binary form and prefixing zeros when necessary to make the total number of digits equal to $(n - 2)$. We define the numbers

$$
\begin{aligned}
a_{rs} &= (s-1)^{\text{st}} \text{ digit (counting from the left) of } r \text{ in binary form} \\
&\qquad [s = 2, 3, \cdots, n-1; r = 0, 1, \cdots, (2^{n-2} - 1)]. \qquad (9) \\
a_{r1} &= 1 \qquad [r = 0, 1, \cdots, (2^{n-2} - 1)].
\end{aligned}
$$

All the 0-chains containing y_1 but not y_n are

$$C_r = \sum_{s=1}^{n-1} a_{rs} y_s , \qquad (r = 0, 1, \cdots, (2^{n-2} - 1). \tag{10}$$

Hence the function f which we shall call the state-function of the graph with respect to nodes y_1 and y_n is

$$f(x_1, x_2, \cdots, x_b) = \prod_{r=0}^{(2^{n-2}-1)} \delta C_r , \tag{11}$$

and is zero if and only if y_1 and y_n are not connected.

By formula (8)

$$\delta C_r = \sum_{s=1}^{n-1} a_{rs} \delta y_s = \sum_{s=1}^{n-1} a_{rs} \left(\sum_{k=1}^{b} A_{ks} x_k \right).$$

Therefore,

$$\delta C_r = \sum_{k=1}^{b} \left(\sum_{s=1}^{n-1} a_{rs} A_{ks} \right) x_k ,$$

or

$$\delta C_r = \sum_{k=1}^{b} \lambda_{rk} x_k ,$$

where

$$\lambda_{rk} = \sum_{s=1}^{n-1} a_{rs} A_{ks} , \qquad [r = 0, 1, \cdots, (2^{n-2} - 1)\ k = 1, 2, \cdots, b]. \tag{12}$$

Thus in terms of the incidence matrix the state function f can be written

$$f(x_1, x_2, \cdots, x_b) = \prod_{r=0}^{(2^{n-2}-1)} \left(\sum_{k=1}^{b} \lambda_{rk} x_k \right). \tag{13}$$

For the circuit in Fig. 1 considered as a circuit between y_1 and y_4, the numbers in binary form from zero to $2^{n-2} - 1 = 2^2 - 1 = 3$ are 00, 01, 10, and 11. Hence the numbers a_{rs} are given by

$$(a_{rs}) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and the zero chains including y_1 but not y_4 are

$$C_0 = y_1 ,$$

$$C_1 = y_1 + y_3 ,$$

$$C_2 = y_1 + y_2 ,$$

$$C_3 = y_1 + y_2 + y_3 .$$

The coboundaries are

$$\delta C_0 = x_1 + x_4 ,$$

$$\delta C_1 = x_1 + x_2 + x_5 ,$$

$$\delta C_2 = x_2 + x_3 + x_4 ,$$

$$\delta C_3 = x_3 + x_5 ,$$

and the state-function f is

$$f(x_1 , x_2 , \cdots , x_5) = (x_1 + x_4)(x_1 + x_2 + x_5)(x_2 + x_3 + x_4)(x_3 + x_5).$$

Precisely the same state-function except for the order of the factors is obtained by excluding y_1 and including y_n in all the zero chains. This can be seen from the fact that the desired 0-chains can be obtained by adding $y_1 + y_n$ to the given chains, since by the modulo-two addition this replaces y_1 by y_n . The coboundaries of these chains are, therefore, obtained from the coboundaries of the given chains by adding $\delta(y_1 + y_n)$. Since the sum of all the nodes is a cocycle we have $\delta(y_1 + y_n) = \delta(\sum_{k=2}^{n-1} y_k)$. Hence we can get the desired coboundaries from the given ones by adding $\sum_{k=2}^{n-1} y_k$ to the given 0-chains instead of $y_1 + y_n$. The numbers a_{rs} which we get by this computation are obtained from the binary representation of r by adding one to each of the $(n - 2)$ rightmost digits of r modulo-two. The resulting sums must be distinct since if they were not, the original numbers would not have been distinct. Since there are $2^{n-2} - 1$ of these, this list must be a permutation of the original list and hence the new (λ_{rs}) matrix is a row-permutation of the original and we get the same state-function.

We define a subgraph of a given graph as the graph obtained by deleting any branches (but not the nodes of these branches) from the given graph; and we have

Theorem II. The state-function f of a subgraph with respect to a given pair of nodes is obtained from the state-function f of the given graph with respect to the same nodes by setting all the branches not in the subgraph equal to zero (or deleting them) in the state-function of the given graph.

The incidence matrix of the subgraph is obtained from the incidence matrix of the graph for the boundary operator by replacing the ones in the rows corresponding to the deleted branches by zeros, and the coboundaries of the nodes with respect to the sub-

graph are obtained from the coboundaries of the nodes with respect to the graph by setting them equal to zero (or deleting them) in the latter. Hence the coboundaries of the 0-chains C_r with respect to the subgraph are obtained by setting the deleted branches of the graph equal to zero in the coboundaries of the 0-chains of the original graph. Consequently by the definition of the state-function, the theorem is proved.

In the graph of Fig. 1, for example, if we delete branches x_1 and x_4 the incidence matrix α_s of the subgraph becomes

$$\bar{A}_{rs} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the coboundaries become

$$\begin{aligned} \delta C_0 &= 0, \\ \delta C_1 &= x_2 + x_5 \,, \\ \delta C_2 &= x_2 + x_3 \,, \\ \delta C_3 &= x_3 + x_5 \,; \end{aligned}$$

and the state-function becomes

$$f_s\,(x_2\,, x_3\,, x_5) = f(0, x_2\,, x_3\,, 0, x_5) = 0,$$

and we see that nodes y_1 and y_4 are not connected.

This interpretation of the formula can be stated as

Theorem III. The connection between two nodes is zero when a given set of branches is opened (deleted) if and only if this set contains all the branches in at least one of the 0-chain coboundaries. Conversely, the given nodes are connected if and only if at least one branch in each of the 0-chain coboundaries is not opened provided the state-function is not identically zero.

This theorem shows how to write a state-function from the conditions of operation.

We define equivalent state-functions as functions which are zero for precisely the same sets of branch states and Theorem III has as a consequence a minimization method, namely,

Theorem IV. If the coboundary of any 0-chain, i.e., factor of f, is contained in another factor then the state function obtained from the given state-function by deleting the first factor is equivalent to the given state-function.

It is clear that the results of combinatorial switching theory can be recovered from this point of view.

As the last two examples we take branches in series and parallel.

For the series network of Fig. 2(a) the above method yields $f(x_1\,, x_2) = x_1x_2$ and for the parallel network of Fig. 2(b), $f(x_1\,, x_2) = x_1 + x_2$. As in the method of Boolean

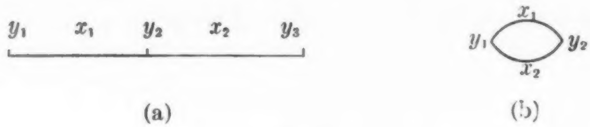

FIG. 2.

algebra, if we consider the states of the branches as being determined by individual state-functions, we can write the state functions of series-parallel networks directly by inspection.

Thus, in addition to the explicit formulas for the states of general networks the preceding theorems show that the theory of contact networks can be founded on an algebraic topological basis. It is not suggested that this approach be a substitute for the methods of Shannon but rather a supplement, and in the next section we consider the conversion of these formulas into the formulas of Boolean algebra.

4. The reduction of state formulas of Boolean algebra. Following Shannon (l.c.) we introduce the Boolean elements 0^* and 1^* where

$$0^* \oplus 0^* = 0^*,$$

$$0^* \oplus 1^* = 1^* \oplus 0^* = 1^* \oplus 1^* = 1^*,$$

$$0^* \cdot 0^* = 0^* \cdot 1^* = 1^* \cdot 0^* = 0^*,$$

$$1^* \cdot 1^* = 1^*.$$

The admittance between two nodes is 1^* if the circuit is closed and 0^* if the circuit is open. Also if β is a Boolean variable which assumes the value 1^* when a given relay is operated and 0^* otherwise, then the states of the normally open contacts of the relay are denoted by β and the states of the normally closed contacts are denoted by the complement of β, i.e., $\bar{\beta}$. If we denote the state of the branch x_k of the graph of the contact network by α_k then the state-function is transformed into a Boolean function and we note that the value of this Boolean function is 0^*, if and only if at least one factor is 0^*. A factor is 0^* if and only if each of its terms is 0^*. By comparison with the results of the preceding section we see that the value of this Boolean function is 0^* if and only if the circuit between the given nodes is open. Consequently, if the circuit is closed each factor has the value 1^*, and the value of the Boolean function is 1^*. This completes the proof of

Theorem V. The Boolean admittance function for a pair of terminals of a network is obtained from the state-function with respect to those terminals by replacing the branches by their admittances.

The calculations are simplified if the columns of the incidence matrix A are used as column vectors representing the coboundaries.

It may also be noted that the admittances between different pairs of nodes of a network are related in a simple way so that if we have one state-function we can get the others immediately. This can be seen from the following considerations. Assume that the factors of the state-function with respect to terminals y_1 and y_n are known. Assume that these have been obtained by including y_1 in all the 0-chains since we get the same results by using either y_1 or y_n. To compute the state-function between y_1 and y_k we

need the 0-chains which include y_1 but not y_k, i.e., we must replace y_k in the first set of 0-chains by y_n. Since y_k occurs in C_r multiplied by a_{rk}, the replacement can be effected by adding $a_{rk}(y_k + y_n)$ to C_r. Hence the factors of the desired state-function are obtained by adding $a_{rk}(\delta y_k + \delta y_n)$ to δC_r. A similar argument holds for replacing node y_1.

Theorem VI. The state-function for the nodes y_i and y_j can be obtained from the state-function for the nodes y_i and y_k by adding $a_{rk}(\delta y_k + \delta y_j)$ to the factors δC_r of the given state-function.

The matrix of these state-functions when the branches are replaced by their admittances is the admittance matrix of Lunts [7] if the diagonal elements are taken as 1*.

5. The synthesis problem for a two-terminal network. A method for finding general networks, i.e., not necessarily series-parallel networks, which realizes the graph of a contact network for a given two-terminal Boolean admittance will be given in this section. We proceed by reversing the analysis in Secs. 3 and 4. The first step is to write the given function as a product of sums of admittances. The second step is to assign branches to each admittance. At this stage there may be a considerable number of possibilities. One extreme is to assign a different branch to each appearance of a contact admittance or its complement; the other extreme is to assign the same branch to each appearance of a variable. The third step consists of writing a state-function by replacing the admittance variables by the corresponding branch variables. The fourth step is to select an integer n such that $b + 1$ is greater than n and 2^{n-2} is not less than the number of factors of the state-function.

In the fifth step if 2^{n-2} is greater than the number of factors of the state-function, then factors subject only to the restriction that the terms of such a factor contain at least the set of nodes in one of the given factors are added so the total number of factors is 2^{n-2}. The sixth step is to assign a number r $[r = 0, 1, 2, \cdots (2^{n-2} - 1)]$ to each of the factors. In the seventh step the equations for the elements of the incidence matrix are written using Eq. (12). The last step is to solve these equations. If the solution has the property that there are no more than two non-zero elements in any row of the solution displayed in matrix form, i.e., is a partial incidence matrix, a column is added so that an element of this column is 0 if there are two ones or no ones in this row, and 1 if there is only one 1 in this row. If the partial incidence matrix does not satisfy the above condition then the computations can be repeated for possibly different assignments of r, b, n, and correspondences of the admittances and branches. An example will be given below. That the above problem has at least one solution is obvious since a series-parallel network can always be drawn.

To clarify the method we retrieve the bridge network of Sec. 1 where $f(\alpha_1, \alpha_2, \cdots, \alpha_5) = (\alpha_1 + \alpha_4)(\alpha_1 + \alpha_2 + \alpha_5)(\alpha_2 + \alpha_3 + \alpha_4)(\alpha_3 + \alpha_5)$. We choose $n = 4$, $b = 5$ and replace α_k by x_k. The numbers $r = 0, 1, 2, 3$ are assigned so that

$$\delta C_0 = x_1 + x_4$$

$$\delta C_1 = x_1 + x_2 + x_5$$

$$\delta C_2 = x_2 + x_3 + x_4$$

$$\delta C_3 = x_3 + x_5 .$$

Thus the matrices A and λ are given by

$$a = (a_{rs}) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \lambda = (\lambda_{rk}) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

and the system of equations (12) becomes

$$\begin{aligned} A_{k1} &= \lambda_{0k} , \\ A_{k1} + A_{k3} &= \lambda_{1k} , \\ A_{k1} + A_{k2} &= \lambda_{2k} , \\ A_{k1} + A_{k2} + A_{k3} &= \lambda_{3k} . \end{aligned}$$
$$(k = 1, 2, 3, 4, 5)$$

The solution of this system is

$$\begin{aligned} A_{k1} &= \lambda_{0k} , \\ A_{k2} &= \lambda_{0k} + \lambda_{2k} , \\ A_{k3} &= \lambda_{0k} + \lambda_{1k} , \end{aligned}$$

which in matrix form is the partial incidence matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

We get the incidence matrix by adding a column which brings the number of ones in each row to two,

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which is the original matrix. The contact network can be drawn immediately.

The equations for the general case may be restated as

$$\sum_{s=1}^{n-1} a_{rs}\delta y_s = \delta C_r \qquad [r = 0, 1, \cdots, (2^{n-2} - 1)]. \tag{14}$$

From the definition of a_{rs} we see that if the system has a solution it must be

$$\begin{aligned}
\delta y_1 &= \delta C_0 , \\
\delta y_2 &= \delta C_0 + \delta C_{2^{n-3}} , \\
&\cdots\cdots\cdots\cdots \\
\delta y_k &= \delta C_0 + \delta C_{2^{n-k-1}} , \\
&\cdots\cdots\cdots\cdots \\
\delta y_{n-1} &= \delta C_0 + \delta C_1 .
\end{aligned}$$

We call these chains a reduced set of factors. Since the ordering of the factors in the state-function is at our disposal we may determine solutions by choosing one of the factors as δC_0 and adding this factor to each of the others. If from these $(2^{n-2} - 1)$ chains we can select $(n - 2)$ chains such that no branch occurs more than twice as a term in the set of chains consisting of these $(n - 2)$ chains together with the chain designated as δC_0, then we have a partial incidence matrix. The consistency of the set of equations (14) must of course be verified by substitution of the trial solution.

Thus for a given assignment of branches, nodes, and order of factors the synthesis problem for a given state-function can be solved if and only if the system of equations (14) is consistent and the set of coboundaries (15) satisfies the condition that no branch appears more than twice.

The consistency equations (14) may be regarded from another point of view. If we add δC_0 (mod 2) to each of the equations of (14), the resulting system is generated from the $(n - 2)$ sets δy_2 , δy_3 , $\cdots$ δy_{n-1} by taking all possible sums, i.e., the set $(\delta C_0 + \delta C_k)$ $[k = 0, 1, 2, \cdots (2^{n-2} - 1)]$ is isomorphic to a Boolean algebra with $(n - 2)$ atoms. The determination of whether or not this is the case can be shown to be independent of the choice of δC_0 , and since the order of the factors of the state-function can be assigned arbitrarily, the consistency can also be shown to be independent of the ordering of the factors. Thus the test of consistency requires only that an arbitrarily selected factor be added to all the factors and any $(n - 1)$ of these sums (modulo 2) be chosen as atoms. If each of these sums is a linear combination of the atoms then the set of equations (14) is consistent, otherwise the system is inconsistent and the assignment of branches to the admittances must be made in another way.

If the number of factors of the state-function is less than 2^{n-2} then the additional factors have the admittance one when the circuit is closed. Thus these factors must arise from branches whose admittances are complementary or must contain as a subset the branches which are present in another factor. This can be verified by generating the complete reduced set $(\delta C_r + \delta C_0)$ $(r = 1, 2, \cdots, 2^{n-2} - 1)$ adding δC_0 and verifying the above condition. It follows that for a given assignment of branches and choice of n the verification of the consistency equations (14) can be made independently of the choice of δC_0 , i.e., holds for all choices, and when the consistency equations are satisfied the problem is reduced to determining whether an assignment of the factors of the state-function can be made so that each branch occurs no more than twice in these factors. This may not be the case even for a consistent set.

It may be remarked that the complexity of this method is due in part to the fact that all circuits realizing the given function can be obtained this way.

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BOOK REVIEWS

(*Continued from p. 172*)

Fundamentals of gas dynamics. Edited by H. W. Emmons (Volume III of high speed aerodynamics and jet propulsion). Princeton University Press, 1958. xiii + 749 pp. $20.00.

It is a pleasure to record the appearance of another volume of this monumental series. The title is perhaps not too informative, and the first duty of the reviewer is to give a short list of contents and authors: The equations of gas dynamics (Tsien); One-dimensional treatment of steady gas dynamics (Crocco); One-dimensional treatment of nonsteady gas dynamics (Kantrowitz); The basic theory of gasdynamic discontinuities (Hayes); Shock wave interactions (Polachek and Seeger); Condensation phenomena in high speed flows (Stever); Introduction to combustion (v. Kármán); Gas dynamics of flame fronts (Emmons); Gas dynamics of detonations (Taylor and Tankin); Flow of rarified gases (Schaaf and Chambré).

The author of a "handbook" article always has to face the choice between the restricted aim of providing an introduction to his subject and guide to the literature, and the ambitious aim of giving a definitive and exhaustive account. Tsien has chosen the more modest alternative and contributed a lucid description of the equations underlying the "classical" part of Gas Dynamics, with indications of where to look for the "modern" departures. The stress is less on the derivation of the Navier-Stokes equations than on the discussion of the derived systems of general equations, such as the circulation and entropy theorems, the stream-function approach and the variational formulation for potential flow.

Crocco has not only chosen the other alternative, but indeed interpreted the title of his article in its full, literal generality. There are no steady compressible flow processes in ducted machinery so complicated that useful one-dimensional models could not be inferred from a physical understanding of the essential mechanisms—and here the models are displayed in a sweep never attempted before, over the whole gamut from the ideal nozzle to the ducted rocket motor. The article is a book in its own right, with a number of original papers thrown in for good measure, and one surmises that only illness prevented Professor Crocco from adding a handbook of chemical engineering! The approach is broad and versatile—physical discussion of complex viscous, thermodynamical and chemical interactions alternate with engineering charts—and the number of topics caught in the Crocco net defies enumeration here. No doubt, this article by itself will ensure a heavy demand for the volume in the aeronautical, mechanical and chemical engineering industries.

A particular trait of the "one-dimensional" theory has always been that its conceptual basis is vague, and even this problem is attacked in the article. The starting point is the recognition that uniformity of physical quantities, or even physical mechanisms, over any cross-section of the duct is not a necessary premise. Large parts of the theory deal with the interaction between quite different streams flowing side by side. The outline of a general mathematical theory of the application of the conservation principles to mean values representing such flows is sketched and elaborated in the direction of a classification of processes.

Kantrowitz' choice, on the other hand, is a very personal introduction to the propagation of plane-wave pulses and the stability of transonic duct flow. The treatment is physically illuminating, even if it lacks the unambiguous lucidity of Tsien. The article also contains a brief review of the procedures of the numerical method of characteristics and a sketch of a few results relating to strong shock propagation.

After the leisurely style of Kantrowitz, Hayes' telegram on the theory of shock, detonation and deflagration fronts comes indeed as a shock. The first part is devoted to a thorough discussion of the system of equations representing the conservation principles and the additional relations which make the system determinate—one might call it the general theory of the Hugoniot curve. In the second part, the ordinary differential equation governing the structure of the front on the basis of the Navier-Stokes equations for one-dimensional steady motion is discussed, both for the shock and the front with one-parameter exothermic reaction. For weak shocks, the description is extended to the case of slow variation in time. As with all telegrams, the onus of deciphering is on the reader—the reviewer passes on with the feeling that his appreciation of this article is rather incomplete.

The discussion of the systems of equations governing the reflection and refraction of shocks follows

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FOLDING OF A LAYERED VISCOELASTIC MEDIUM DERIVED FROM AN EXACT STABILITY THEORY OF A CONTINUUM UNDER INITIAL STRESS*

By

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Abstract. A layer of viscoelastic material embedded in an infinite medium also viscoelastic tends to fold through instability when the system is compressed in a direction parallel with the layer. This phenomenon which was treated previously by using an approximate plate theory, [1, 2], is analyzed here by using an exact theory for the deformation of a continuum under prestress [3, 4, 5, 6]. The effect of the compressive prestress in the embedding medium is taken into account, and it is found that although it is not always negligible, it tends to be very small under conditions where the instability of the layer is strong. The same conclusion holds for the error involved in the use of a plate theory for the layer instead of the exact equations for a prestressed continuum. In the course of the analysis we have also treated the problem of a semi-infinite viscoelastic half space subject to a uniform internal compression parallel with the boundary and a surface load normal to this boundary. The compressive load produces an increase of the surface deflection under the normal load. This effect appears through an amplification factor which is evaluated numerically for the particular example of an elastic body. It is shown that under certain conditions the free surface of the compressed semi-infinite medium may become unstable and will tend to wrinkle. This is suggested as a probable explanation for the wrinkles which appear on the surface of a body subjected to a plastic compression.

1. Introduction. We consider a layer of viscoelastic material surrounded by an unbounded medium also viscoelastic, the whole system being subject to compressive stresses parallel with the direction of the layer. Such stresses may be set up for instance if the whole system is subject to a uniform compressive strain parallel with the layer.

As another example we have the case where a compressive stress in the layer alone will be produced if a swelling of the layer occurs while longitudinal expansion is prevented. The question which we are concerned with here is the stability of the layer under such a system of prestress and the prediction of the deformation of this sytem. We have already analyzed this problem by introducing certain approximations [1, 2]. In particular we have assumed that the layer behaves like a plate, i.e., that it obeys equations of flexure of the type used in strength of materials theory and generalized to apply to viscoelastic media. Another assumption introduced in this previous work is that the prestress in the surrounding medium has a negligible effect.

Our purpose is to check the accuracy of the previous simplified theories by solving the problem from the exact equations for small deformations of a continuum under prestress. General equations for incremental deformations of a body under prestress were derived by this writer some years ago in a series of publications [3, 4, 5, 6]. While the theory was developed in the particular context of elastic continua, the equations

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are applicable to any type of incremental deformations of elastic or anelastic bodies. This is due to the fact that in the development of the theory the physics is separated from the geometry of the problem, and the incremental stress strain relations are left arbitrary. For the reader's convenience, we have rederived the equations in Sec. 2 for the particular case of plane strain. In Sec. 3 these equations are first applied to the evaluation of the surface deflection under a normal load for a semi-infinite viscoelastic medium, which is at the same time subject to a compression in a direction parallel with the boundary.

The relations between the incremental stresses and strains are assumed to be the same as those derived by the present writer from thermodynamics [7]. The nature of the approximation which may be involved in this assumption is discussed in a previous publication [2]. It is found that the lateral compression produces an amplification of the surface deflection and its magnitude is evaluated. The possibility of an instability of the free surface under a lateral compression is analyzed in Sec. 4. It is pointed out that although the magnitude of this instability is in general quite small, the effect may be large for incremental deformations in the plastic range where the so-called tangent modulus is the significant parameter. The instability should manifest itself in the form of a wrinkling of the surface, a phenomenon which is actually observed on the surface of a compressed solid in the plastic range.

In Sec. 5 a complete analysis is given for the folding of a layer of uniform thickness embedded in a medium of infinite extent. We have assumed that there is no friction at the interface of the layer and the medium. This affords a direct comparison with the results obtained in [1] and [2] by an approximate theory and with the same assumption of perfect interfacial slip.* Results of the previous section are used in order to evaluate the influence of the compressive stress in the surrounding medium. It is found that in general this compression may be neglected, as it does not affect substantially the wave length of the folding. The case where the layer and the surrounding medium are both incompressible viscous fluids is investigated in detail.

The rate of growth of the amplitude of folding as a function of the wave length has been derived. The dominant wave length—i.e., that which exhibits the maximum rate of amplitude growth—is evaluated as a function of the viscosity ratio of the two media. This dominant wave length is independent of the load. Results are compared with the results of the approximate theory of [1]. It is found that this approximate theory agrees very well with the present results when the layer viscosity is larger than about seventy times the viscosity of the surrounding medium. This is also the condition for the magnitude of the instability to become significant so that the folding amplitude increases at a rate fast enough to be observable in practice.

2. The theory of deformation of a prestressed solid. We shall first recall briefly the theory of incremental stress and deformation of a solid under initial stress. We assume the incremental deformation to be small so that the theory is linear. Such a theory was developed by the writer [3, 4, 5, 6] for the case of an elastic body. However, it is independent of the physical nature of the body and is therefore applicable to viscoelastic media. In order to simplify the presentation, we shall limit ourselves here to a two dimensional deformation.

We are dealing with the problem of a continuum undergoing small deformation

*An evaluation of the effect of interfacial adherence has been given in [2].

from an initial state which is not one of zero stress. The initial stress field is defined by components S_{11}, $S_{12} = S_{21}$, S_{22}, referred to x, y axes. This being a two dimensional case, we assume that the other components of the initial stress field vanish or do not appear in the equations. The condition of equilibrium of this stress field is

$$\begin{aligned} \frac{\partial S_{11}}{\partial x} + \frac{\partial S_{12}}{\partial y} + \rho X = 0, \\ \frac{\partial S_{21}}{\partial x} + \frac{\partial S_{22}}{\partial y} + \rho Y = 0, \end{aligned} \tag{2.1}$$

with a mass density ρ and a body force X, Y per unit mass. The problem is to formulate the laws of deformation of this medium where small incremental stresses and strains are superimposed on the initial state.

In treating this problem we have departed from the traditional viewpoint of the mathematician and defined the incremental tensors in a different way. Let us first consider the strain. A point P of initial coordinates x, y in the prestressed state is displaced to P' of coordinates

$$\begin{aligned} \xi = x + u, \\ \eta = y + v. \end{aligned} \tag{2.2}$$

In this transformation the material around P undergoes a rotation

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{2.3}$$

and a deformation

$$e_{xx} = \frac{\partial u}{\partial x} \qquad e_{yy} = \frac{\partial v}{\partial y} \qquad e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \tag{2.4}$$

In order to deal with the problem of relating stress to strain, it is important to consider the phenomenon from the view point of an observer rotating with the material through an angle ω. If we denote by E_{ij} the strain components referred to rotated axes, then it is clear that to the first order these are the same as e_{ij} defined above,

$$E_{ij} = e_{ij} . \tag{2.5}$$

This is not true, however, for the incremental stress. If we refer the stress to axes rotating with the material, the total stress acting on the element is

$$\begin{Bmatrix} S_{11} + s_{11} & S_{12} + s_{12} \\ S_{21} + s_{21} & S_{22} + s_{22} \end{Bmatrix}. \tag{2.6}$$

The incremental stress s_{ij} is referred to rotated axes and depends only on the strain e_{ij}. The nature of this relation is a physical problem which we shall consider in the next section. We shall deal here only with the geometrical aspects of the problem and established equations for the incremental stress field s_{ij} which express the condition of equilibrium of the field. This is done by first referring the stress field to the unrotated initial directions x, y. These stress components are to the first order

$$
\begin{aligned}
\sigma_{xx} &= S_{11} + s_{11} - 2S_{12}\omega, \\
\sigma_{yy} &= S_{22} + s_{22} + 2S_{12}\omega, \\
\sigma_{xy} &= S_{12} + s_{12} + (S_{11} - S_{22})\omega.
\end{aligned}
\tag{2.7}
$$

Note that in the deformation this stress field is the one at point $\xi = x + u$ and $\eta = y + \sigma$. Also that the mass density has now become ρ' by the incremental volume change.* The equations of equilibrium of the field σ_{ij} are

$$
\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial \xi} + \frac{\partial \sigma_{xy}}{\partial \eta} + \rho' X &= 0, \\
\frac{\partial \sigma_{xy}}{\partial \xi} + \frac{\partial \sigma_{yy}}{\partial \eta} + \rho' Y &= 0.
\end{aligned}
\tag{2.8}
$$

We now express these equations by means of the original coordinates x, y. Using the relations

$$
\begin{aligned}
d\xi &= \left(1 + \frac{\partial u}{\partial x}\right) dx + \frac{\partial u}{\partial y}\, dy \\
d\eta &= \frac{\partial v}{\partial x}\, dx + \left(1 + \frac{\partial v}{\partial y}\right) dy
\end{aligned}
\tag{2.9}
$$

and solving for dx, dy, we find the partial derivatives of x, y with respect to ξ,η, e.g.

$$
\frac{\partial x}{\partial \xi} = \frac{1}{D}\left(1 + \frac{\partial v}{\partial y}\right) \quad \text{etc.} \tag{2.10}
$$

D is the Jacobian, i.e., the determinant of the transformation (2.9). With these expressions we may write

$$
\frac{\partial \sigma_{xx}}{\partial \xi} = \frac{\partial \sigma_{xx}}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \sigma_{xx}}{\partial y}\frac{\partial y}{\partial \xi} = \frac{1}{D}\frac{\partial \sigma_{xx}}{\partial x}\left(1 + \frac{\partial v}{\partial y}\right) - \frac{1}{D}\frac{\partial \sigma_{xx}}{\partial y}\frac{\partial v}{\partial x}. \tag{2.11}
$$

Conservation of mass requires

$$
D\rho' = \rho. \tag{2.12}
$$

The equilibrium equations (2.8) may then be expressed in terms of x and y. Substituting the values (2.7) for σ_{ij} retaining only first order terms, gives

$$
\begin{aligned}
&\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - 2\frac{\partial}{\partial x}(S_{12}\omega) + \frac{\partial}{\partial y}[(S_{11} - S_{22})\omega] \\
&\qquad + \frac{\partial S_{11}}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial S_{11}}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial S_{12}}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial S_{12}}{\partial y}\frac{\partial u}{\partial x} = 0.
\end{aligned}
\tag{2.13}
$$

Another equation is obtained by permutation of the symbols and changing ω in $-\omega$. These equations are further simplified by introducing the following identities

$$
\begin{aligned}
\frac{\partial v}{\partial x} &= e_{xy} + \omega \\
\frac{\partial u}{\partial y} &= e_{xy} - \omega
\end{aligned}
\tag{2.14}
$$

*For simplicity we assume here that the body force is independent of the coordinates. The more general case is considered in [4].

and by taking into account the equilibrium condition (2.1) for the initial stress field. Equations (2.13) are then written

$$\begin{aligned}\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \rho Y\omega - 2S_{12}\frac{\partial \omega}{\partial x} + (S_{11} - S_{22})\frac{\partial \omega}{\partial y}& \\ + \frac{\partial S_{11}}{\partial x} e_{yy} - \left(\frac{\partial S_{11}}{\partial y} + \frac{\partial S_{12}}{\partial x}\right) e_{xy} + \frac{\partial S_{12}}{\partial y} e_{xx} &= 0.\end{aligned} \tag{2.15}$$

The other equation is also obtained by permutation of the variables and changing ω in $-\omega$. If the initial field is uniform, the terms written on the second line disappear and the equations become

$$\begin{aligned}\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \rho Y\omega - 2S_{12}\frac{\partial \omega}{\partial x} + (S_{11} - S_{22})\frac{\partial \omega}{\partial y} &= 0, \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - \rho X\omega + 2S_{12}\frac{\partial \omega}{\partial y} + (S_{11} - S_{22})\frac{\partial \omega}{\partial x} &= 0.\end{aligned} \tag{2.16}$$

It is interesting to see that in the absence of a body force the only additional terms due to the pre-stress depend on the total initial shear and disappear if the initial stress is hydrostatic.

The boundary conditions are found from the stresses σ_{ij}. The force acting on an arc element $d\xi$, $d\eta$, after deformation is

$$\begin{aligned}dF_x &= \sigma_{xx}\, d\eta - \sigma_{xy}\, d\xi, \\ dF_y &= \sigma_{xy}\, d\eta - \sigma_{yy}\, d\xi.\end{aligned} \tag{2.17}$$

The force F is acting on matter lying to the left side of the arc element $d\xi$, $d\eta$ in a counter-clockwise coordinate system and to the right in a clockwise system. Substituting the values (2.7) for the stresses, expressions (2.9) for $d\xi$, $d\eta$ and retaining only first order terms, we find

$$\begin{aligned}dF_x = &-[s_{12} + S_{12} - S_{22}\omega + S_{12}e_{xx} - S_{11}e_{xy}]\, dx \\ &+ [s_{11} + S_{11} - S_{12}\omega + S_{11}e_{yy} - S_{12}e_{xy}]\, dy, \\ dF_y = &-[s_{22} + S_{22} + S_{12}\omega + S_{22}e_{xx} - S_{12}e_{xy}]\, dx \\ &+ [s_{12} + S_{12} + S_{11}\omega + S_{12}e_{yy} - S_{22}e_{xy}]\, dy.\end{aligned} \tag{2.18}$$

This gives the force acting on an element of arc initially of components dx, dy, at point x, y.

3. The viscoelastic half space under combined surface loading and lateral compression. We shall apply the previous equations to the problem of determining the deflection of the surface of a viscoelastic half space under the action of a load normal to the surface and a condition of prestress which is a compression in a direction parallel to the surface (Fig. 1).

The surface is along the x-axis and the y-axis is directed inward. The initial stress system is assumed to be

$$S_{22} = S_{12} = 0, \qquad S_{11} = -P, \tag{3.1}$$

i.e., a uniform compression along the x-direction. We assume the body force to be zero. Equations (2.16) become

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P\frac{\partial \omega}{\partial y} = 0,$$
$$\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P\frac{\partial \omega}{\partial x} = 0. \tag{3.2}$$

The next step is now to introduce a relation between the incremental stresses s_{ij} and the incremental strain e_{ij}. Such relations have not yet been developed in the most general case for a pre-stressed material, and some difficulty arises in this connection. We have shown [4] that for an elastic body under prestress, the elastic constants cannot be the same as in the unstressed condition unless the prestress system is hydrostatic. We therefore point out that, in general, it must be considered an approximation to use—as we shall do here—the same stress-strain relations for the prestressed and the initially stress-free medium. We have discussed this point more extensively in [2].

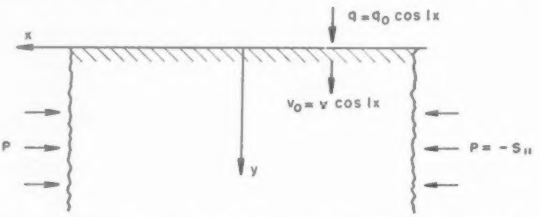

FIG. 1. Viscoelastic half space under combined surface loads and lateral compression.

Another point arises if the body is viscous for slow deformations. Then the initial state is one of steady strain rate, and there are additional incremental strains which are not due to the incremental stresses s_{ij} alone.

In the applications below we shall, therefore, consider the total displacement field as $u_i + u_i^*$ where u_i^* is the displacement associated with the steady strain rate due to the initial stress S_{ij}. The term u_i then represents the departure of the displacement from the initial uniform strain rate. The incremental strain and rotation appearing in equations (2.15), (2.16) and (2.18) should then be written as $e_{ij} + e_{ij}^*$ and $\omega + \omega^*$ where e_{ij}^* and ω^* correspond to the initial steady state.

In the applications below, the components e_{ij}^* and ω^* drop out of the equations and do not appear in the formulation of the problem. We may, therefore, disregard the term u_i^* in the present treatment. In general, of course, since e_{ij}^* and ω^* are linear functions of time, they will disappear from the equations by introducing second time derivatives of all variables.

The operational stress-strain relations for an initially stress-free medium were established in [7] and found to be

$$s_{ij} = 2\bar{Q}e_{ij} + \delta_{ij}\bar{R}e \tag{3.3}$$

with the unit matrix δ_{ij} and $e = e_{xx} + e_{yy}$.

The operators are

$$\bar{Q} = p\int_0^\infty \frac{Q(r)}{p + r}\gamma(r)\,dr + Q + Q'p, \tag{3.4}$$

$$\bar{R} = p\int_0^\infty \frac{R(r)}{p + r}\gamma(r)\,dr + R + R'p, \tag{3.5}$$

where p designates the time differential operator

$$p = \frac{d}{dt}. \tag{3.6}$$

By a well-known property of such operators, the relations are also valid if p is a real or a complex quantity, provided all variables contain the exponential factor e^{pt}.

Substituting expressions (3.3) for the stress into the equations (3.2), we find

$$\begin{aligned} \bar{Q}\nabla^2 u + [\bar{Q} + \bar{R}] \frac{\partial e}{\partial x} - P \frac{\partial \omega}{\partial y} &= 0, \\ \bar{Q}\nabla^2 v + [\bar{Q} + \bar{R}] \frac{\partial e}{\partial y} - P \frac{\partial \omega}{\partial x} &= 0. \end{aligned} \tag{3.7}$$

It can be verified that solutions of these equations with a sinusoidal factor along x are

$$\begin{aligned} u &= -\sin lx[Ae^{-ly} + Ck(1 - \beta)e^{-lky}], \\ v &= -\cos lx[Ae^{-ly} + C(1 - \beta k^2)e^{-lky}]. \end{aligned} \tag{3.8}$$

These solutions are analogous to those developed in [3, 6] for the purely elastic case. In these expressions A and C are unknown operators, and we have put

$$2\beta = \frac{2\bar{Q} + P}{2\bar{Q} + \bar{R}}, \qquad k = \left(\frac{1 - \zeta}{1 + \zeta}\right)^{1/2}, \qquad \zeta = \frac{P}{2\bar{Q}}. \tag{3.9}$$

If k is chosen positive the solutions (3.8) represent deformations confined to a region near the surface. The same is true if k is complex and chosen such that its real part is positive.

Consider now the case where a *normal load* of sinusoidal distribution is applied to the surface ($y = 0$). We wish to determine the deflection at the surface due to the simultaneous action of the surface load and the compressive preload P in the material. The surface load will be represented as

$$\frac{dF_y}{dx} = q = q_0 \cos lx. \tag{3.10}$$

It is acting downward in the y direction as indicated in Fig. 1. The tangential component of the boundary force vanishes

$$dF_x = 0. \tag{3.11}$$

Introducing expressions (2.18) into the relations (3.10) (3.11), we obtain the boundary conditions

$$\begin{aligned} e_{xy} &= 0, \\ s_{22} &= -q_0 \cos l_x = 2\bar{Q}e_{yy} + \bar{R}e. \end{aligned} \tag{3.12}$$

Substituting in these conditions the values u and v from expressions (3.8) and putting $y = 0$ yields

$$\begin{aligned} A + \frac{C}{2}(1 + k^2 - 2\beta k^2) &= 0, \\ A + Ck(1 - \beta k^2) + \frac{C\bar{R}}{2\bar{Q}} k\beta(1 - k^2) &= -\frac{q}{2\bar{Q}l}. \end{aligned} \tag{3.13}$$

The conditions determine the operators A and C. The deflection of the surface is also sinusoidal and represented by

$$v_0 = V \cos lx. \tag{3.14}$$

The ratio of the load q to the deflection v_0 at the surface $y = 0$ is then given by

$$\frac{q_0}{Vl} = \frac{q}{v_0 l} = \frac{1}{2} B\left[\frac{k(1 + \zeta)^2 - 1}{\zeta} - \alpha\zeta\right] \tag{3.15}$$

with the notation

$$B = \frac{4\bar{Q}(\bar{Q} + \bar{R})}{2\bar{Q} + \bar{R}}, \qquad \alpha = \frac{\bar{Q}}{\bar{Q} + \bar{R}}. \tag{3.16}$$

The effect of the compressive load P arises through the bracketed factor in Eq. (3.15). This factor is

$$\varphi = \frac{1}{\zeta}[k(1 + \zeta)^2 - 1] - \alpha\zeta. \tag{3.17}$$

The deflection under a load q_0 is

$$V = \frac{2}{lB\varphi} q_0 . \tag{3.18}$$

If a constant sinusoidally distributed load is suddenly applied at time $t = 0$ with a maximum value unity,

$$q_0 = 1(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{3.19}$$

it may be represented by the complex integral

$$q_0 = 1(t) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{e^{pt}}{p}\, dp. \tag{3.20}$$

Hence the deflection under this load is

$$V(t) = \frac{1}{il\pi}\int_{C-i\infty}^{C+i\infty} \frac{e^{pt}}{pB\varphi}\, dp. \tag{3.21}$$

In this expression $\bar{B}$ and φ are functions of p.

The effect of the lateral compression P appears through the factor φ. If this lateral compression is zero, then

$$\zeta = 0, \qquad \varphi = 1 \tag{3.22}$$

and the deflection is

$$V(t) = \frac{1}{il\pi}\int_{C-i\infty}^{C+i\infty} \frac{e^{pt}}{pB}\, dp. \tag{3.23}$$

In order to give an idea of the magnitudes involved, let us consider first the case of a

purely elastic incompressible material. Incompressibility corresponds to

$$\bar{R} = \infty \quad \text{or} \quad \alpha = 0. \tag{3.24}$$

With a shear modulus G, we put

$$\bar{Q} = G. \tag{3.25}$$

Hence

$$\varphi = \frac{k(1 + \zeta)^2 - 1}{\zeta} \tag{3.26}$$

with

$$\zeta = P/2G.$$

The deflection is amplified by the lateral compression P, and the amplification factor is $1/\varphi$. Values of this amplification factor as a function of ζ are given in Table 1.

TABLE 1.

ζ	$1/\varphi$
0.50	1.67
0.60	2.14
0.70	3.29
0.80	10.00
0.85	∞

We see that when $\zeta = 0.85$ i.e. for $P = 1.7G$ the amplification factor is infinite and the surface becomes *unstable*. We also notice that the amplification factor becomes appreciable only for values $\zeta > 0.5$; hence, for compressive loads P larger than G these loads are of the order of the elastic modulus. In a material of constant modulus, they can only occur for loads corresponding to a very large compressive strain. However, we must remember that the present theory is for incremental deformations. Hence, the modulus G appearing in the above expression is actually the so-called *tangent modulus*. In many cases where plastic deformations are involved, this modulus decreases with the load, and the above values may well fall within observable ranges.

In regard to the significance of the amplification factor, it should be remarked that it is the same for all wave lengths. Hence, the deflection of the surface under an arbitrary load distribution may be derived from the deflection under the same surface load and zero compression ($P = 0$) by simply multiplying the latter by the amplification factor. The compressive load introduces no distortion of the deflection but simply a change of magnitude.

It can easily be seen that the amplification effect of the compressive load on the surface deflection of a half space also applies when the initial deflection is not due to a surface loading but due to irregularities and waviness. This departure from perfect

smoothness is amplified without distortion by the compressive loads. This suggests that under high compressive loads, surfaces which appear to be smooth initially will tend to wrinkle. Since the amplification factor is dependent only on the tangent modulus, this mechanism suggests itself particularly as the explanation for the wrinkling which appears on surfaces of materials compressed in the plastic range.

4. Instability of the free surface of a half space under compression. Let us now examine the condition of instability of the free surface of a half space under compression. This corresponds to the existence of solutions different from zero for the deflection V while the surface load q_0 is zero. Equation (3.18) may be written

$$q_0 = \tfrac{1}{2}B\varphi Vl. \tag{4.1}$$

The condition $q_0 = 0$ with $V \neq 0$ amounts to

$$\varphi = \frac{1}{\zeta}[k(1+\zeta)^2 - 1] - \alpha\zeta = 0 \tag{4.2}$$

which may also be written

$$\frac{k(1+\zeta)^2}{\zeta} = \frac{1+\alpha\zeta^2}{\zeta} \tag{4.3}$$

or

$$2 - 2\zeta^2 - \zeta^3 = 2\alpha\zeta + \alpha^2\zeta^3. \tag{4.4}$$

Let us first examine the elastic case. The operators $\bar{Q}$ and $\bar{R}$ are replaced by the Lamé constants

$$\bar{Q} = G \qquad \bar{R} = \lambda. \tag{4.5}$$

Poisson's ratio being designated by ν we find

$$\alpha = \frac{G}{\lambda + G} = 1 - 2\nu. \tag{4.6}$$

Also

$$\zeta = P/2G. \tag{4.7}$$

The critical value of ζ is the real root of Eq. (4.4). This root depends on Poisson's ratio only and its value is shown in Table 2.

TABLE 2

ν	ζ
0	.55
$\frac{1}{4}$	.68
$\frac{1}{2}$	.84

The instability occurs at values of the compressive load P comparable with the shear modulus G. However, as already pointed out above, since the latter is the tangent modulus, its value may be low enough in the plastic range for the surface instability to occur within the practical range of P.

As an example of a viscous material, let us now consider the case of a viscous incompressible fluid. Some clarification is needed here as to what is meant by instability. We assume that the half space is initially under the influence of a compressive load P parallel with the free surface and that the fluid is initially in a steady state of uniform rate of deformation under this load. All velocities and strain rates are initially constant in time and space. The stability of the free surface under those conditions is determined by considering the possible exponential growth of a disturbance starting from a certain instantaneous configuration considered as the initial state.* The possibility of such instability is easily determined from the above analysis. For an incompressible fluid, the operators become

$$\bar{Q} = \mu p, \qquad \bar{R} = \infty, \tag{4.8}$$

with μ the viscosity coefficient. Since the condition $\bar{R} = \infty$ implies $\alpha = 0$, Eq. (4.4) becomes

$$2 - 2\zeta^2 - \zeta^3 = 0 \tag{4.9}$$

with

$$\zeta = \frac{P}{2\bar{Q}} = \frac{P}{2\mu p}.$$

Equation (4.9) is a cubic in the unknown p. For values of p which are roots of this equation, there exist disturbances proportional to the factor e^{pt}. Instability corresponds to roots which are real and positive or complex with a positive real part. The three roots p of (4.9) are

$$\begin{aligned} p_1 &= 1.192 \frac{P}{2\mu}, \\ p_2 &= (-0.596 + 0.255i) \frac{P}{2\mu}, \\ p_3 &= (-0.596 - 0.255i) \frac{P}{2\mu}. \end{aligned} \tag{4.10}$$

When substituted in Eq. (4.2), i.e. $\varphi = 0$, the complex roots do not furnish a solution if we choose k to have a positive real part. Hence they do not correspond to the physical problem.

The root p_1 corresponds to instability. In order to appreciate the magnitude of this instability, let us consider a time t_1 during which the undisturbed uniform flow produces a 25% compressive strain. The rate of strain in the x direction under the load P is

$$\dot{e}_{xx} = -\frac{0.25}{t_1} = -\frac{P}{4\mu}. \tag{4.11}$$

Hence,

$$t_1 = \frac{\mu}{P}, \qquad p_1 t_1 = 0.596. \tag{4.12}$$

---

*In dealing with a viscous fluid whose initial state of stress gives rise to a steady strain rate, we must refer to the remark Sec. 3 by which it is justified to consider not the actual incremental deformation, but the departure of the deformation from the steady state itself.

This means that in the time interval t_1 during which the material is squeezed by 25%, any disturbance of the free surface from a perfect plane is amplified by a factor $e^{p_1 t_1} = 1.81$. This instability is of such a mild form that it will be very difficult to observe in practice.*

A similar procedure may be used for any kind of viscoelastic half space by solving Eq. (4.4) as an equation for p with the operators $\bar{Q}$ and $\bar{R}$ as appropriate functions of the unknown p.

5. Characteristic equation for the stability of the embedded layer. We consider a layer of thickness h embedded in an infinite medium. The layer and the surrounding medium have different viscoelastic properties. The layer is subject to an initial compression P and the surrounding medium to a compression P_1 (Fig. 2). This initial state

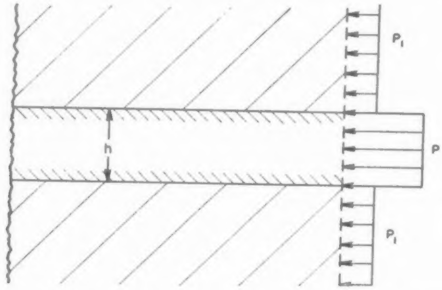

FIG. 2. Layer embedded in infinite medium under lateral compression.

of stress will depend on the type of initial deformation imposed on the system and on the viscoelastic parameters of the two media. For example, if the two media are incompressible viscous fluids and if the system is squeezed uniformly in a direction parallel with the layer, the compressive stresses are in the ratio

$$\frac{P}{P_1} = \frac{\mu}{\mu_1}, \tag{5.1}$$

where μ is the viscosity coefficient of the layer and μ_1 that of the surrounding medium. The stress could also be due to other causes, as, for instance, an increase of volume of the layer while longitudinal elongation is prevented. In this case the stress P_1 in the surrounding medium can be zero.

In order to simplify the formulation, we shall assume that the interface between the layer and the medium is perfectly lubricated so that there are no tangential stresses at the interface. A previously developed approximate theory for the case of perfect and imperfect adherence [2] indicates that this assumption does not modify the result substantially. The properties of the layer are represented by the operators $\bar{Q}$ and $\bar{R}$ and that of the surrounding medium by $\bar{Q}_1$ and $\bar{R}_1$. We consider the layer separately and locate the x-axis in the plane equidistant from the two faces (Fig. 3). The y-axis is normal to the layer. Flexure of the layer is represented by antisymmetric solutions of Eqs.

*Since the theory is only valid for small deformations, its application to the case of a large compressive strain of 25 per cent will, of course, give only an indication of the orders of magnitude of the amplification effect.

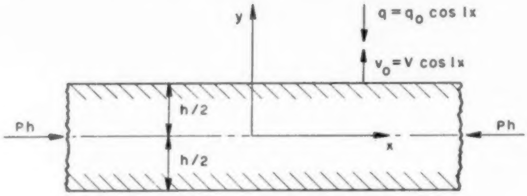

FIG. 3. Forces acting on the layer.

(3.7). They are obtained by adding two solutions of the type (3.8) by using both positive and negative exponents. We find

$$u = -\sin lx[A \sinh ly + Ck(1 - \beta) \sinh lky],$$
$$v = \cos lx[A \cosh ly + C(1 - \beta k^2) \cosh lky], \tag{5.2}$$

where β and k are defined by (3.9).

This also coincides with the solution obtained previously for the purely elastic case [3]. The normal force applied to the layer at the boundary $y = h/2$ and due to the surrounding medium is

$$q = q_0 \cos lx. \tag{5.3}$$

This load is taken positive in the *negative* y direction, and the tangential force vanishes owing to the condition of perfect slip at the interface. Because of symmetry, boundary conditions imposed at one interface are automatically satisfied at the other. We shall consider the interface at $y = h/2$. Applying (2.18) we find the boundary conditions

$$\begin{aligned} \frac{dF_x}{dx} &= -s_{12} - Pe_{xy} = 0 \qquad y = \frac{h}{2} \\ \frac{dF_y}{dx} &= -s_{22} = -q_0 \cos lx. \end{aligned} \tag{5.4}$$

From expressions (3.3) for s_{12} and s_{22} in terms of the strains we find at $y = h/2$,

$$\begin{aligned} e_{xy} &= 0, \\ 2\bar{Q}e_{yy} + \bar{R}(e_{xx} + e_{yy}) &= -q_0 \cos lx. \end{aligned} \tag{5.5}$$

Substituting the expressions (5.2) for u and v yields two equations for the operators A and C in terms of q_0. We must also consider the boundary value of the normal displacement v. This is expressed by substituting $y = h/2$ in the second equation (5.2). We find

$$v = V \cos lx = \left[A \cosh \frac{lh}{2} + C(1 - \beta k^2) \cosh \frac{lkh}{2}\right] \cos lx. \tag{5.6}$$

The normal deflection at the boundary $y = h/2$ is $V \cos lx$. By substituting in (5.6) the expressions for A and C determined from the two equations (5.5), we find a relation between the load q_0 and the interface deflection V. This relation may be written

$$\frac{2q_0}{lVB} = \frac{1}{\zeta}(1 + \alpha\zeta^2) \tanh \gamma - \frac{(1 + \zeta)^2}{\zeta} k \tanh k\gamma, \tag{5.7}$$

where

$$\gamma = \tfrac{1}{2}lh, \qquad \zeta = \frac{P}{2\bar{Q}},$$

and B and α are operators depending solely on the viscoelastic properties of the layer

$$\alpha = \frac{\bar{Q}}{\bar{Q} + \bar{R}}, \qquad B = \frac{4\bar{Q}(\bar{Q} + \bar{R})}{2\bar{Q} + \bar{R}}. \tag{5.8}$$

At this point we still have to match boundary conditions for the layer and the half space representing the embedding medium. This is readily done if we consider the ratio of the load q_0 to deflection V for the surface of the half space which was obtained in Sec. 3 and given by expression

$$\frac{2q_0}{lV} = B_1\varphi_1, \tag{5.9}$$

where

$$B_1 = \frac{4\bar{Q}_1(\bar{Q}_1 + \bar{R}_1)}{2\bar{Q}_1 + \bar{R}_1}, \qquad \varphi_1 = \frac{2 - 2\zeta_1^2 - \zeta_1^3}{k_1(1 + \zeta_1)^2 + 1} - \alpha_1\zeta_1, \tag{5.10}$$

$$\zeta_1 = \frac{P_1}{2\bar{Q}_1} \qquad k_1 = \left(\frac{1 - \zeta_1}{1 + \zeta_1}\right)^{1/2} \qquad \alpha_1 = \frac{\bar{Q}_1}{\bar{Q}_1 + \bar{R}_1}.$$

The quantities B_1 and α_1 depend only on the viscoelastic properties of the surrounding medium, while the others include the value of the prestress P_1 in this medium. This prestress is contained in the expression φ_1 and, as was pointed out above, we may consider $1/\varphi_1$ as an amplification factor for the deflection at the surface of a half space. The value of the amplification factor is unity when the prestress is zero ($P_1 = \zeta_1 = 0$). Since the load q_0 and the normal deflection V are the same for both the layer and the medium at the interface matching of the boundary, conditions amount to equating the values of q_0/lV appearing on the left side of the two equations (5.7) and (5.9). This yields the characteristic equation

$$\frac{B_1}{B}\varphi_1 = \frac{1}{\zeta}(1 + \alpha\zeta^2)\tanh\gamma - \frac{(1 + \zeta)^2}{\zeta}k\tanh k\gamma. \tag{5.11}$$

It is true that the values of q_0 and V for the layer and the medium are not those of the same location of the interface because of the relative slip of the two media. However, the error is of the second order and is irrelevant in a first order theory.

The parameters of the surrounding medium are contained solely in $B_1\varphi_1$ on the left hand side. The significance of this characteristic equation lies in the fact that for given materials and a given prestress field P and P_1, it constitutes a relation between the nondimensional wave number $\gamma = \frac{1}{2}lh$ of the folding and the value of p. This value of p is the coefficient in the time factor e^{pt} which multiplies the amplitude of the folding for

any particular wave number γ initially present as a disturbance in the layer. In a system with random initial disturbance, those wave lengths will appear for which p has the largest positive value. We shall disregard the possibility that there are complex values for p as encountered above in the case of the viscoelastic half space and assume that such complex values are not physically significant.

6. Numerical results and discussion. An approximate expression for the characteristic equation (5.11) is obtained if we expand the hyperbolic tangents in power series to the third power in γ. We find

$$\frac{B_1}{B}\varphi_1 = \zeta\gamma(1 + \alpha) - \tfrac{1}{3}\gamma^3[2 - (1 - \alpha)\zeta]. \tag{6.1}$$

If the wave length of the folding is sufficiently large, γ and ζ are small. We may further neglect $(1 - \alpha)\zeta$ in the bracket. Also we may assume ζ_1 to be small, which amounts to neglecting the prestress P_1 in the surrounding medium. Hence we may put $\varphi_1 = 1$. Equation (6.1) then reduces to

$$\frac{B_1}{B} = \zeta\gamma(1 + \alpha) - \tfrac{2}{3}\gamma^3. \tag{6.2}$$

This may also be written

$$P = \frac{1}{12}B(lh)^2 + \frac{B_1}{lh}. \tag{6.3}$$

This last equation coincides with Eq. (3.1) of [1] obtained by neglecting the prestress P_1 in the surrounding medium and using a "thin plate" type theory for the layer. A complete discussion of the approximate equation (6.3) was given in [1] for the general case of two media of arbitrary viscoelastic properties and for a certain number of particular cases.

The exact equation (5.11) provides a means to test the accuracy of the approximate theory. For this purpose it is sufficient to compare the two theories on a simple example. We shall choose the case of two incompressible fluids. Incompressible fluids are represented by

$$\begin{aligned} \bar{R} &= \bar{R}_1 = \infty, \\ \alpha &= \alpha_1 = 0, \\ \bar{Q} &= \mu p, \\ \bar{Q}_1 &= \mu_1 p, \end{aligned} \tag{6.4}$$

where μ is the viscosity coefficient of the layer and μ_1 that of the surrounding medium. Introducing these expressions into the exact characteristic equation (5.11), we derive

$$\frac{\mu_1}{\mu}\varphi_1 = \frac{1}{\zeta}\tanh\gamma - \frac{(1 + \zeta)^2}{\zeta}k\tanh k\gamma, \tag{6.5}$$

where

$$\begin{aligned} \varphi_1 &= \frac{2 - 2\zeta_1^2 - \zeta_1^3}{k_1(1 + \zeta_1)^2 + 1}, \\ \zeta &= \frac{P}{2\mu p}, \qquad \zeta_1 = \frac{P_1}{2\mu_1 p}. \end{aligned} \tag{6.6}$$

There are two types of approximations involved in Eq. (6.2): that due to the use of thin plate theory and the other due to neglecting the prestress P_1 of the surrounding medium. Let us discuss the effect of each separately. Therefore we put $\varphi_1 = 1$ in Eq. (6.5), i.e., we consider first the error due to the use of plate theory alone. The characteristic equation is then

$$\frac{\mu_1}{\mu} = \frac{1}{\zeta} \tanh \gamma - \frac{(1 + \zeta)^2}{\zeta} k \tanh k\gamma. \tag{6.7}$$

This is a relation between these variables representing a function ζ of γ with a parameter μ/μ_1. The variable ζ versus γ is plotted in Figs. (4) and (5) for six values of μ/μ_1.

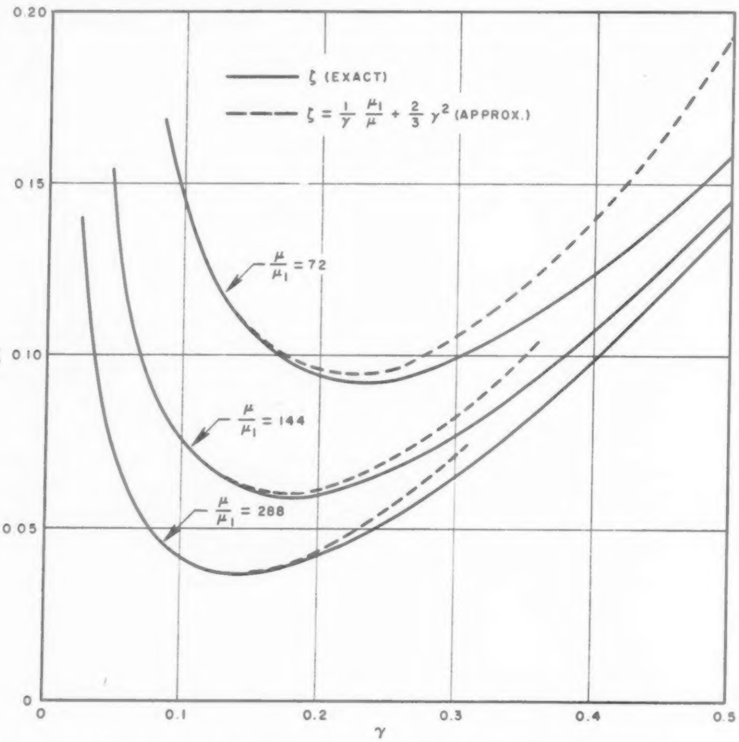

FIG. 4. Plot of $\zeta = P/2\mu p$ versus γ and the parameter μ/μ_1 from the exact relation (6.7) and the approximate solution (6.9).

$$\mu/\mu_1 = 6, 12, 36, 72, 144, 288.$$

Consider now the approximate equation (6.2) which corresponds to the thin plate theory and, also, to neglect of the prestress P_1. For incompressible fluids, this equation becomes

$$\frac{\mu_1}{\mu} = \zeta\gamma - \tfrac{2}{3}\gamma^3 \tag{6.8}$$

or

$$\zeta = \frac{1}{\gamma}\frac{\mu_1}{\mu} + \tfrac{2}{3}\gamma^2. \tag{6.9}$$

This also represents a family of functions ζ of γ which are plotted as dotted lines in Figs. (4) and (5) for the same values of μ/μ_1. Examination of the two plots affords an immediate comparison between the exact and approximate theories. Of particular interest is the minimum value of ζ denoted by $\zeta_{\min}$ and the value γ at which this minimum occurs. It is easily seen that the minimum of ζ for a given value of the compressive load P

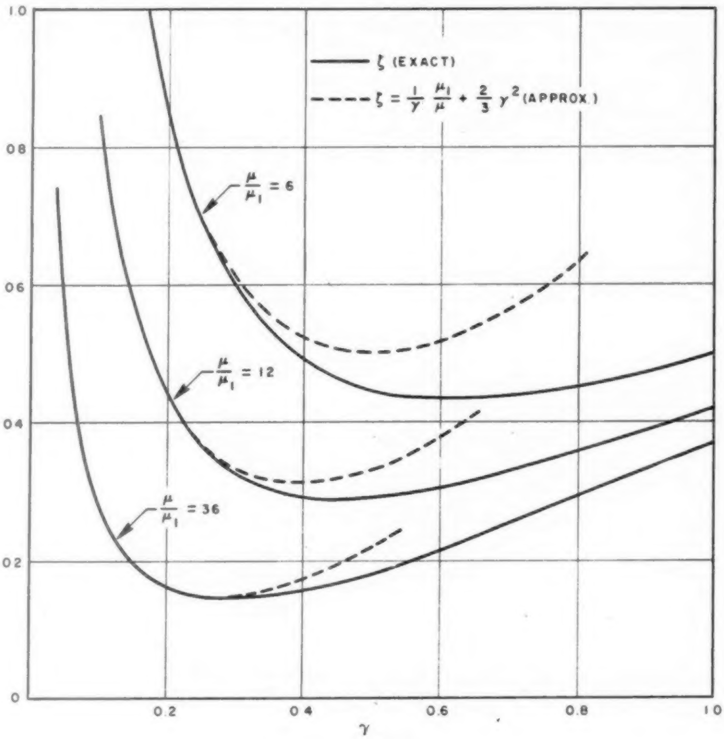

FIG. 5. Same as Fig. 4.

corresponds to maximum values of p. The significance of a root p of Eq. (6.7) is given by the fact that for such roots a disturbance of wave number l of the layer increases proportionally to the factor e^{pt}. Hence, maximum values of p correspond to wave lengths of maximum rate of growth. We have referred to this wave length as the *dominant wave length.* The wave number l_d of this dominant wave length is therefore determined by

$$\gamma_d = \tfrac{1}{2}l_d h = \pi\frac{h}{L_d} \tag{6.10}$$

(L_d = dominant wave length).

From formula (6.9) we find the approximate values

$$\gamma_d = \left(\frac{3}{4}\frac{\mu_1}{\mu}\right)^{1/3}, \tag{6.11}$$

$$\zeta_{\min} = 2\left(\frac{3}{4}\frac{\mu_1}{\mu}\right)^{2/3}. \tag{6.12}$$

The approximate value (6.11) of γ_d as a function of $(\mu_1/\mu)^{1/3}$ is shown in Fig. 6 by the dotted straight line, while the exact value of γ_d [determined from Eq. (6.7)] is represented by the full line. Comparison between the two curves indicates that a discrepancy in the value of γ_d begins to be noticeable for values $(\mu_1/\mu)^{1/3} > 1/4$ or $\mu/\mu_1 < 64$. The error in the dominant wave length may rise to 10–15% when μ/μ_1 becomes of the order of ten.

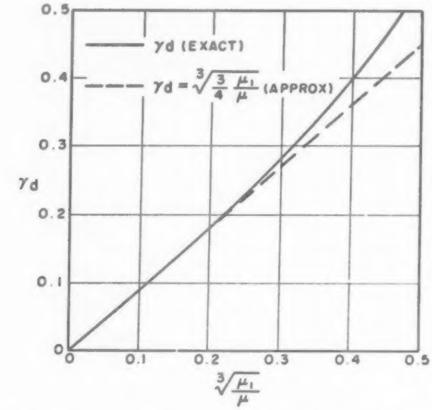

FIG. 6. Value $\gamma_d = \pi h/L_d$ (L_d = dominant wavelength) as a function of μ/μ_1 by exact and approximate theory.

We have also plotted the exact value of $\zeta_{\min}$ versus $(\mu_1/\mu)^{2/3}$ as determined from Eq. (6.7). It is represented by the full line in Fig. 7. The approximate value (6.12) is represented by a dotted straight line. The error becomes noticeable for $(\mu_1/\mu)^{2/3} > 1/16$; hence, again for $\mu/\mu_1 < 64$ For smaller values of μ/μ_1 , the error may reach 10–15%.

Let us now look upon the other approximation involved in Eq. (6.9) and which is due to neglect of the prestress P_1 in the surrounding medium. In order to evaluate the effect of this prestress, we must reintroduce the factor φ_1 which we had put equal to unity. This amounts to using the same graphs as in Figs. 4 to 7, but with a corrected value $\mu_1\varphi_1$ instead of μ_1 for the viscosity of the surrounding medium. Let us evaluate the value of the correction factor φ_1 in the vicinity of the dominant wave number (near $\zeta_{\min}$). Assuming the layered material to be squeezed uniformly so that the strains are the same in the layer and the surrounding medium the compressive prestresses P_1 and P satisfy relation (5.1) and we derive

$$\zeta = \zeta_1 . \tag{6.13}$$

We may, therefore, evaluate φ_1 as a function of ζ. Values of φ_1 are given in Table 3.

TABLE 3.

ζ	φ_1	$(\varphi_1)^{\frac{1}{3}}$	$\varphi_1^{2/3}$
.1	.946	.980	.962
.3	.804	.930	.865
.5	.598	.841	.710

The values of φ_1 show a decrease in the "effective value" $\mu_1\varphi_1$ of the viscosity coefficient of the surrounding medium due to the prestress. The decrease is about 5% or less for values of $\zeta_{\min} < 0.1$, i.e., for $\mu/\mu_1 > 64$ and may be as high as 40% for the lower values of μ/μ_1. The effect on the dominant wave length is much smaller and is proportional to $(\varphi_1)^{1/3}$. From the values of $(\varphi_1)^{1/3}$ in Table 3 and from Fig. 6, it can be seen that the dominant wave length will be increased from 2 to 16 per cent in the same range. Reference to Fig. 7 and values of $(\varphi_1)^{2/3}$ show that $\zeta_{\min}$ will be decreased by an amount somewhat larger

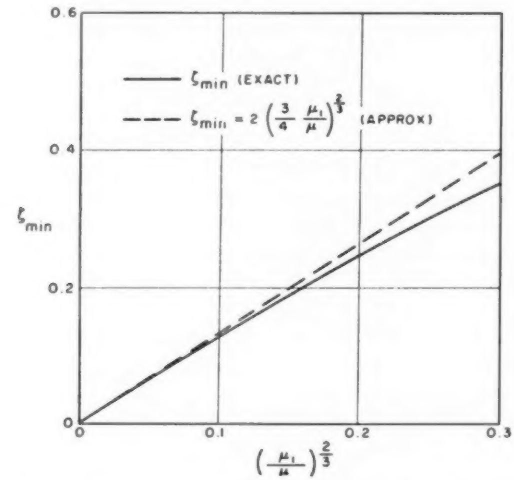

FIG. 7. The minimum value of ζ as a function of μ/μ_1 by exact and approximate theory.

due to the prestress. In general, we may conclude that the effect of the prestress P_1 in the embedding medium becomes negligible if the viscosity ratio μ/μ_1 of the layer to the medium is larger than about 64. It is interesting to note that this was also found to be the condition for the error, due to thin plate theory, to be negligible.

In order to complete the present discussion, we must also pay attention to the degree of instability involved in the phenomenon of folding. The dominant wave length determined by γ_d is that for which an initial waviness in the layer increases at the fastest rate. After a time t, the amplitude is multiplied by the factor e^{pt} where p is given by

$$p = \frac{P}{2\mu\zeta_{\min}}. \tag{6.14}$$

We may ask, for instance, what is the value of the factor e^{pt} after a time t_1 such that

the layered material is squeezed by 25 per cent under the compressive stresses P and P_1.

Referring to the same argument as in Sec. 4 in the case of the half space, we find

$$t_1 = \frac{\mu}{P} \tag{6.15}$$

and

$$pt_1 = 1/2\zeta_{\min} .$$

From Fig. 7 giving $\zeta_{\min}$ as a function of μ/μ_1, we may derive the amplification factor e^{pt_1} in terms of μ/μ_1. Values are shown in Table 4.

TABLE 4.

μ/μ_1	$\exp(pt_1)$
8	3.78
27	14.8
64	127
125	1940

The amplification factor increases sharply and becomes quite large for values $\mu/\mu_1 > 64$. Below these values the magnitude of the amplification factor becomes relatively small, indicating that the instability in that range is not too significant and will not exhibit very sharp features. This lower range of values is also the one in which the errors become appreciable if we apply thin plate theory and neglect the prestress in the surrounding medium. We may conclude that in the region of significant instability Eq. (6.2) of the approximate theory is applicable.

The present discussion is limited to the case of incompressible viscous fluids. A similar discussion can be carried out for any two materials of arbitrary viscoelastic properties represented by four suitable operators. If the two materials are incompressible, the plots in Figs. 4 to 7 represent master graphs by which the folding may be analyzed quantitatively by means of very simple calculations.

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THERMAL SHOCK IN AN ELASTIC BODY WITH A SPHERICAL CAVITY*

BY

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Summary. This investigation aims at the dynamic thermoelastic response of an infinite medium with a spherical cavity to a sudden uniform change in the temperature of its internal boundary. By means of the Laplace transform, a closed solution to this problem—exact within classical elastokinetics—is obtained in terms of error functions of real and complex arguments. The ensuing temperature stresses are compared with the corresponding quasi-static results.

Introduction. The conventional quasi-static treatment of transient thermoelastic problems rests on the assumption, apparently introduced first by Duhamel [1][1], that the inertia terms may be neglected in the governing field equations. The quality of the approximate solutions thus obtained evidently depends both upon the magnitude of the time-gradients inherent in the temperature field and upon the size of the relevant intrinsic inertia parameters.

Duhamel's hypothesis has been the object of several recent studies. The earliest investigation of this kind is due to Danilovskaya [2] (1950), who determined the dynamic thermal stresses induced in an elastic half-space by a sudden uniform heating (or cooling) of the entire boundary, if the body is constrained to a uniaxial motion perpendicular to the bounding plane[2]. Later on, Danilovskaya [4] generalized her solution to accommodate convective boundary conditions. Still more recently, the present authors [5] established explicitly the thermal displacements associated with the stresses given in [2], [3] and extended the completed solution to the case of a gradual (ramp-type) change of the surface-temperature. Related studies in the theory of beams and plates were carried out by Boley [6] and by Boley and Barber [7]. Finally, we mention thermoelastic investigations by Nowacki [8] and Ignaczak [9] in which the inertia terms are taken into account, although neither of these papers is concerned with a quantitative assessment of the dynamic effects thus arising.

As was pointed out in [5], the problem treated in [2], [3], [5] suffers from two deficiencies as a vehicle for examining the adequacy of the traditional quasi-static methods in transient thermoelasticity. First, the artificial constraint of the medium imposed in [2] results in a severe degeneracy of the corresponding quasi-static stress distribution, the normal stress perpendicular to the boundary being identically zero. Second, the absence of a characteristic length in this particular problem causes the pertinent inertia parameter to be hidden in the dimensionless time and distance scales whence its influence upon the dynamic effects sought, is obscured.

For the preceding reasons it appears desirable to examine the inertia effects en-

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[1]Numbers in brackets refer to the list of references at the end of the paper.

[2]The results contained in [2] were independently reached by Mura [3] (1952).

countered in a less degenerate thermoelastic space problem. This leads us to consider an infinite, homogeneous and isotropic, elastic medium with a spherical cavity, the boundary of which is exposed to an instantaneous change of temperature. We assume that the body, which is free from loading, is initially at rest, and seek the ensuing thermal stresses and displacements if the temperature field obeys the (uncoupled) heat-conduction equation. The corresponding *quasi-static* solution was given and discussed in [10]. It should be mentioned that the present *dynamic* thermoelastic problem is related to an ordinary problem of elastokinetics, i.e. to the determination of the stresses produced by a sudden uniform pressure applied to the surface of a spherical cavity in an infinite elastic medium. An exact closed solution to this more elementary problem was obtained first by Jeffreys [11] (1931) and has since been rediscovered by various subsequent authors.

Formulation of problem. Dimensionless variables. Let (r, θ, φ) respectively denote the radial coordinate, the co-latitude, and the longitude of a spherical coordinate system. Suppose that the medium under consideration occupies the region $a \leq r < \infty$, where a is the radius of the spherical cavity. The thermoelastic problem described in the Introduction is characterized by polar symmetry about the origin, whence

$$T = T(r, t); \qquad u_r = u(r, t), \qquad u_\theta = u_\varphi = 0, \tag{1}$$

in which t is the time, T the temperature, while $(u_r, u_\theta, u_\varphi)$ designate the spherical components of the displacement vector.

If T_0 is the constant temperature suddenly assumed by the internal boundary $r = a$, and κ stands for the thermal diffusivity of the material, we may introduce a dimensionless radial coordinate, time and temperature, by means of

$$\rho = \frac{r}{a}, \qquad \tau = \frac{\kappa t}{a^2}, \qquad \varphi = \frac{T}{T_0}. \tag{2}$$

With reference to these dimensionless variables, the independent temperature problem is governed by the (uncoupled) heat-conduction equation

$$\frac{\partial^2 \varphi}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial \varphi}{\partial \rho} = \frac{\partial \varphi}{\partial \tau}, \tag{3}$$

subject to the initial condition

$$\varphi(\rho, 0) = 0, \tag{4}$$

together with the boundary and regularity conditions

$$\varphi(1, \tau) = h(\tau), \qquad \varphi(\rho, \tau) \to 0 \quad \text{as} \quad \rho \to \infty, \tag{5}$$

where $h(\tau)$ is the Heaviside step-function, given by

$$h(\tau) = 0 \quad \text{for} \quad -\infty < \tau < 0, \qquad h(\tau) = 1 \quad \text{for} \quad 0 < \tau < \infty. \tag{6}$$

We turn next to the characterization of the thermal displacements and stresses associated with the temperature field $\varphi(\rho, \tau)$. To this end, let a dimensionless radial displacement u^*, as well as dimensionless spherical components of normal stress σ_r^* and σ_θ^*, be defined through

$$\left.\begin{aligned} u^* &= \frac{1-\nu}{(1+\nu)\alpha T_0 a}\, u, \\ \sigma_r^* = \frac{1-\nu}{2(1+\nu)\alpha T_0 \mu}\, \sigma_r\,, &\qquad \sigma_\theta^* = \frac{1-\nu}{2(1+\nu)\alpha T_0\mu}\, \sigma_\theta\,, \end{aligned}\right\} \tag{7}$$

in which σ_r , σ_θ are the corresponding physical components of normal stress, whereas μ, ν, and α denote the shear modulus, Poisson's ratio, and the coefficient of thermal expansion, respectively. In view of (1), the spherical components of shear stress vanish identically and $\sigma_\varphi = \sigma_\theta$. Finally, it is expedient to introduce a dimensionless inertia parameter γ, defined by

$$\gamma = \frac{\kappa}{ca}\,, \tag{8}$$

where c is the velocity of irrotational waves,

$$c^2 = \frac{2(1-\nu)\mu}{(1-2\nu)\beta}\,, \tag{9}$$

β being the mass density of the material.

The displacement equation of motion now takes the form

$$\frac{\partial^2 u^*}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u^*}{\partial \rho} - \frac{2u^*}{\rho^2} = \gamma^2 \frac{\partial^2 u^*}{\partial \tau^2} + \frac{\partial \varphi}{\partial \rho}\,, \tag{10}$$

while the stress-displacement relations become

$$\left.\begin{aligned} (1-2\nu)\sigma_r^* &= (1-\nu)\frac{\partial u^*}{\partial \rho} + 2\nu\frac{u^*}{\rho} - (1-\nu)\varphi, \\ (1-2\nu)\sigma_\theta^* &= \nu\frac{\partial u^*}{\partial \rho} + \frac{u^*}{\rho} - (1-\nu)\varphi. \end{aligned}\right\} \tag{11}$$

Once $\varphi(\rho, \tau)$ is known, the solution of the thermoelastic problem at hand reduces to the determination of a function $u^*(\rho,\tau)$ which satisfies (10), meets the initial conditions

$$u^*(\rho, 0) = 0, \qquad \left[\frac{\partial u^*}{\partial \tau}\right]_{(\rho,0)} = 0, \tag{12}$$

and is such that the stresses (11) conform to the boundary condition

$$\sigma_r^*(1, \tau) = 0, \tag{13}$$

as well as to the regularity conditions

$$\sigma_r^*(\rho, \tau) \to 0, \qquad \sigma_\theta^*(\rho, \tau) \to 0 \quad \text{as} \quad \rho \to \infty. \tag{14}$$

We observe that the quasi-static formulation of the present problem[3] is obtained by setting γ equal to zero in (10) and by deleting (12).

Introduction of the Laplace transform. Solution in the transform domain. The solution of the elementary heat-conduction problem governed by (3), (4), (5), is well known[4] and admits the closed representation

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[3]See [10].

[4]See, for example, [12], p. 209.

$$\varphi(\rho, \tau) = \frac{1}{\rho}\,\psi(\xi), \qquad \xi = \frac{\rho - 1}{2(\tau)^{1/2}}, \tag{15}$$

where

$$\psi(\xi) \equiv \operatorname{erfc}(\xi) = \frac{2}{\pi^{1/2}} \int_\xi^\infty \exp(-\eta^2)\, d\eta \tag{16}$$

is the complementary error function. For future convenience we recall[5] here that

$$\psi(0) = 1, \qquad \psi(\xi) \to 0 \quad \text{as} \quad \xi \to \infty, \qquad \psi(-\xi) = 2 - \psi(\xi), \tag{17}$$

and cite the power-series development

$$\psi(\xi) = 1 - \frac{2}{\pi^{1/2}} \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2n+1}}{(2n+1)\, \underline{|n}}, \tag{18}$$

as well as the semi-convergent expansion, valid as $\operatorname{Re}(\xi) \to +\infty$,

$$\psi(\xi) = \frac{\exp(-\xi^2)}{\pi^{1/2}\xi} \left[\sum_{n=0}^{N} \frac{(-1)^n \, \underline{|2n}}{\underline{|n}\, (2\xi)^{2n}} + O(\xi^{-2N-3}) \right] \qquad (N = 1, 2, \cdots). \tag{19}$$

With a view toward the determination of the thermal displacements and stresses arising from the temperature distribution (15), let

$$U(\rho, s) = L\{u^*(\rho, \tau)\} = \int_0^\infty \exp(-s\tau) u^*(\rho, \tau)\, d\tau, \tag{20}$$

so that $U(\rho, s)$ is the Laplace transform with respect to τ of $u^*(\rho, \tau)$, s being the transform parameter. From (15),

$$\Phi(\rho, s) = L\{\varphi(\rho, \tau)\} = \frac{1}{\rho s} \exp[-(\rho - 1)s^{1/2}], \tag{21}$$ [6]

whence applying the transform to (10), and bearing in mind (12), we reach the ordinary differential equation

$$\frac{d^2U}{d\rho^2} + \frac{2}{\rho}\frac{dU}{d\rho} - \left(\frac{2}{\rho^2} + \gamma^2 s^2\right) U = -\left(\frac{1}{\rho\, s^{1/2}} + \frac{1}{\rho^2 s}\right) \exp[-(\rho - 1)s^{1/2}]. \tag{22}$$

Furthermore, by virtue of (11) to (14), (20), (21), we arrive at the transformed boundary and regularity conditions

$$\left.\begin{aligned} \left[(1 - \nu)\frac{dU}{d\rho} + 2\nu \frac{U}{\rho} - (1 - \nu)\Phi\right]_{(1,s)} &= 0, \\ U(\rho, s) \to 0 \quad \text{as} \quad \rho &\to \infty. \end{aligned}\right\} \tag{23}$$

Equation (22) is reducible to an inhomogeneous modified Bessel equation. Its general solution is given by

$$U = V(\rho, s) + W(\rho, s), \tag{24}$$

[5]See [13], p. 147.
[6]See [14], p. 246.

where $W(\rho, s)$ is a particular integral of the complete equation, and

$$V(\rho, s) = \frac{\lambda_1(s)}{\rho^{1/2}} I_{3/2}(\gamma s\rho) + \frac{\lambda_2(s)}{\rho^{1/2}} K_{3/2}(\gamma s\rho). \tag{25}$$

Here, $\lambda_1(s)$, $\lambda_2(s)$ are arbitrary functions of s, while $I_{3/2}$ and $K_{3/2}$ designate modified Bessel functions of the first and second kind, respectively. As is readily confirmed, W may be taken in the form

$$W(\rho, s) = \frac{1 + \rho s^{1/2}}{\rho^2 s^2(\gamma^2 s - 1)} \exp[-(\rho - 1)s^{1/2}]. \tag{26}$$

On the other hand, (23) to (26), after an elementary computation, yield

$$V(\rho, s) = \frac{(1 + \gamma\rho s)\{\gamma^2 s^2 + 2p[1 + s^{1/2}]\}}{\rho^2 s^2(1 - \gamma^2 s)[(\gamma s + p)^2 + q^2]} \exp[-\gamma(\rho - 1)s], \tag{27}$$

provided

$$p = \frac{1 - 2\nu}{1 - \nu}, \qquad q = \frac{(1 - 2\nu)^{1/2}}{1 - \nu}. \tag{28}$$

Inverse transforms. Solution of problem. We now proceed to establish the desired inverse transform of $U(\rho, s)$. For this purpose we note from (26) that

$$W = W_1(\rho, s)W_2(\rho, s), \tag{29}$$

if

$$W_1 = \rho^{-2}(\gamma^2 s - 1)^{-1}, \qquad W_2 = (s^{-2} + \rho s^{-3/2}) \exp[-(\rho - 1)s^{1/2}]. \tag{30}$$

Moreover, according to known results[7],

$$\left.\begin{aligned} w_1^*(\rho, \tau) &= L^{-1}\{W_1(\rho, s)\} = \frac{1}{\gamma^2\rho^2} \exp(\tau/\gamma^2), \\ w_2^*(\rho, \tau) &= L^{-1}\{W_2(\rho, s)\} = (\rho + 1)(\tau/\pi)^{1/2} \exp(-\xi^2) + \frac{1 - \rho^2}{2} \psi(\xi) + \tau\psi(\xi), \end{aligned}\right\} \tag{31}$$

where ξ and $\psi(\xi)$ are defined in (15), (16). From (29), (31), and the convolution theorem, follows

$$w^*(\rho, \tau) = L^{-1}\{W(\rho, s)\} = \int_0^\tau w_1^*(\rho, \tau - \eta)w_2^*(\rho, \eta)\, d\eta. \tag{32}$$

Substitution of (31) into (32), and repeated use of integration by parts, ultimately lead to

$$\left.\begin{aligned} w^*(\rho, \tau) = \frac{1}{2\rho^2} [(\rho^2 &- 2\tau - 1 - 2\gamma^2)\psi(\xi) \\ &- 2(\rho + 1)(\tau/\pi)^{1/2} \exp(-\xi^2) + \gamma^2\psi_1(\rho, \tau) - \gamma\rho\psi_2(\rho, \tau)]. \end{aligned}\right\} \tag{33}$$

The auxiliary functions ψ_1, ψ_2 appearing in (33) are accounted for by

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[7]See [14], No. 1, p. 229; No. 4, p. 245; No. 7, p. 246.

$$\left.\begin{aligned}\psi_1(\rho,\tau) &= \psi\left(\xi + \frac{\tau^{1/2}}{\gamma}\right)\exp\left(\frac{\tau}{\gamma^2} + \frac{\rho-1}{\gamma}\right)\\ &\qquad + \psi\left(\xi - \frac{\tau^{1/2}}{\gamma}\right)\exp\left(\frac{\tau}{\gamma^2} - \frac{\rho-1}{\gamma}\right) = \gamma\frac{\partial\psi_2}{\partial\rho},\\ \psi_2(\rho,\tau) &= \psi\left(\xi + \frac{\tau^{1/2}}{\gamma}\right)\exp\left(\frac{\tau}{\gamma^2} + \frac{\rho-1}{\gamma}\right)\\ &\qquad - \psi\left(\xi - \frac{\tau^{1/2}}{\gamma}\right)\exp\left(\frac{\tau}{\gamma^2} - \frac{\rho-1}{\gamma}\right).\end{aligned}\right\} \quad (34)^8$$

Our next task consists in finding the inverse Laplace transform of $V(\rho, s)$ in (27). Evidently,

$$v^*(\rho,\tau) = L^{-1}\{V(\rho,s)\} = h(\omega)L^{-1}\{G(\rho,s)\} \tag{35}$$

with

$$\omega = \tau - \gamma(\rho - 1), \tag{36}$$

if h again denotes the Heaviside step-function given by (6), and G is the coefficient of the exponential function in (27). Moreover, a partial fraction decomposition of $G(\rho, s)$ results in

$$G(\rho,s) = \frac{\gamma^2 + 2ps^{-2} + 2ps^{-3/2}}{\gamma q\rho^2(1 + 2\gamma p + 2\gamma^2 p)}\,\mathrm{Re}\left[\frac{F(\rho)}{s-\gamma^{-2}} - \frac{F(\rho)}{s-k}\right], \tag{37}$$

where "Re" stands for "the real part of", k is the complex parameter defined by

$$k = \frac{-p + iq}{\gamma}, \tag{38}$$

and

$$F(\rho) = -q(\gamma + \rho) + i[1 + p(\gamma - 2\gamma\rho - \rho)]. \tag{39}$$

The inverse of $G(\rho, s)$ is readily found with the aid of the available inverse transform

$$L^{-1}\left\{\frac{s^{-3/2}}{s-\lambda}\right\} = \lambda^{-3/2}[1 - \psi(\lambda^{1/2}\tau^{1/2})]\exp(\lambda\tau) - \frac{2}{\lambda}\left(\frac{\tau}{\pi}\right)^{1/2}, \quad (40)^9$$

in which λ is a constant. Thus, (35) and (37) imply

$$\left.\begin{aligned}v^*(\rho,\tau) = -\frac{h(\omega)}{\rho^2}\Big[&\gamma(\gamma+\rho)\exp(\omega/\gamma^2)\\ &+ A\,\mathrm{Re}\,\{F(\rho)[\gamma^3\psi(\omega^{1/2}/\gamma)\exp(\omega/\gamma^2) - k^{-3/2}\psi(k\omega)^{1/2}\exp(k\omega)\\ &+ B\exp(k\omega) + C[\omega + 2(\omega/\pi)^{1/2}] + D]\}\Big].\end{aligned}\right\} \quad (41)^{10}$$

The letters A, B, C, D, appearing in (41), refer to the following auxiliary functions of ν and γ:

[8] Note that the arguments of the complementary error function ψ here are $\xi + \tau^{1/2}/\gamma$ or $\xi - \tau^{1/2}/\gamma$, as indicated.

[9] See [14], No. 2, p. 233.

[10] Here, as well as subsequently, $k^{1/2}$ stands for the principal value of this root.

$$\left.\begin{aligned} A &= \frac{2p}{\gamma q(1 + 2\gamma p + 2\gamma^2 p)}, \\ B = \frac{\gamma^2}{2p} + k^{-2} + k^{-3/2}, &\qquad C = \gamma^2 - k^{-1}, \qquad D = \gamma^4 - k^{-2}. \end{aligned}\right\} \tag{42}$$

From (24), (32), (35), we have

$$u^* = v^*(\rho, \tau) + w^*(\rho, \tau), \tag{43}$$

where v^* and w^* are given by (41) and (33).

The thermal stresses associated with the displacement (43), are obtained by substitution of (43) into (11). If

$$E = \frac{\partial F}{\partial \rho} = -q - i(1 + 2\gamma)p, \tag{44}$$

the results of these lengthy computations become:

$$\left.\begin{aligned} \sigma_r^* = \frac{h(\omega)}{p\rho^3} \Big[& [\rho^2 + 2\gamma p(\gamma + \rho)] \exp(\omega/\gamma^2) \\ & + A \operatorname{Re} \{[F(\rho)(\rho + 2\gamma p) - E\gamma\rho]\gamma^2 \psi(\omega^{1/2}/\gamma) \exp(\omega/\gamma^2) \\ & + [E\rho - F(\rho)(2p + \gamma k\rho)]k^{-3/2}\psi(k^{1/2}\omega^{1/2}) \exp(k\omega) \\ & + B[F(\rho)(2p + \gamma k\rho) - E\rho] \exp(k\omega) \\ & + C[2pF(\rho) - E\rho][\omega + 2(\omega/\pi)^{1/2}] \\ & + F(\rho)(2pD + \gamma\rho C) - DE\rho\} \Big] \\ & + \frac{1}{\rho^3} [(2\tau + 2\gamma^2 + 1 - \rho^2)\psi(\xi) + 2(\rho + 1)(\tau/\pi)^{1/2} \exp(-\xi^2) \\ & - \left(\frac{\rho^2}{2p} + \gamma^2\right)\psi_1(\rho, \tau) + \gamma\rho\psi_2(\rho, \tau)], \end{aligned}\right\} \tag{45}$$

and, setting

$$m = \frac{\nu}{1 - 2\nu} = \frac{q^2}{p^2} - \frac{1}{p}, \tag{46}$$

$$\left.\begin{aligned} \sigma_\theta^* = \frac{h(\omega)}{\rho^3} \Big[& [m\rho^2 - \gamma(\gamma + \rho)] \exp(\omega/\gamma^2) \\ & + A \operatorname{Re} \{[F(\rho)(m\rho - \gamma) - E\gamma m\rho]\gamma^2 \psi(\omega^{1/2}/\gamma) \exp(\omega/\gamma^2) \\ & + [Em\rho - F(\rho)(\gamma mk\rho - 1)k^{-3/2}\psi(k^{1/2}\omega^{1/2}) \exp(k\omega) \\ & + B[F(\rho)(\gamma mk\rho - 1) - Em\rho] \exp(k\omega) \\ & - C[F(\rho) + Em\rho][\omega + 2(\omega/\pi)^{1/2}] \\ & + F(\rho)(C\gamma m\rho - D) - DEm\rho\} \Big] \\ & - \frac{1}{2\rho^3} [(2\tau + 2\gamma^2 + 1 + \rho^2)\psi(\xi) + 2(\rho + 1)(\tau/\pi)^{1/2} \exp(-\xi^2) \\ & - (\gamma^2 - m\rho^2)\psi_1(\rho, \tau) + \gamma\rho\psi_2(\rho, \tau)]. \end{aligned}\right\} \tag{47}$$

This completes the formal solution to the thermoelastic problem at hand. One confirms by direct substitution, with the aid of (17), (18), (19), that the displacement (43) and the stresses (45), (47), indeed satisfy the field equations (10), (11) if $\omega \neq 0$, and conform to conditions (12), (13), (14).

Properties of solution. Quasi-static and steady-state limits. Examining the structure of the solution obtained in the preceding section, we observe that the displacement (43) and the stresses (45), (47), each contain a term which has the step-function $h(\omega)$ as a multiplier. If $\gamma > 0$, these terms reflect the presence of a spherical shock-wave issuing from the boundary $\rho = 1$ and propagating outward with the velocity c of irrotational waves, given by (9). The remaining terms are diffusive in character and correspond to a signal which is instantaneously received throughout the entire medium.

In accordance with (6), (36), we have

$$\tau = \gamma(\rho - 1), \tag{48}$$

provided ρ and τ momentarily designate the dimensionless radius of the wave-front and the dimensionless time at which this radius is attained. While the time and space derivatives of the radial displacement suffer discontinuities at $\tau = \gamma(\rho - 1)$, u^* itself depends continuously on ρ and τ for $1 < \rho < \infty$, $0 < \tau < \infty$, as required by the physical continuity of the material. On the other hand, the stresses exhibit finite jump-discontinuities at the wave-front. Specifically,

$$\left.\begin{aligned} \sigma_r^*[\rho, \gamma(\rho - 1) +] - \sigma_r^*[\rho, \gamma(\rho - 1) -] &= \frac{1 - \nu}{(1 - 2\nu)\rho}, \\ \sigma_\theta^*[\rho, \gamma(\rho - 1) +] - \sigma_\theta^*[\rho, \gamma(\rho - 1) -] &= \frac{\nu}{(1 - 2\nu)\rho}. \end{aligned}\right\} \tag{49}$$

Thus, σ_θ^* remains continuous only if $\nu = 0$; the jumps in both stresses are independent of the inertia parameter γ and become unbounded as $\nu \to 1/2$, if μ remains fixed in this limit.

It is interesting to compare (49) with the stress-discontinuities generated in Jeffreys' problem [11] by a radial pressure of magnitude σ_0, which is suddenly applied to the spherical boundary and is steadily maintained thereafter. Taking, in this instance, $\tau = ct/a$, $\sigma_r^* = \sigma_r/\sigma_0$, $\sigma_\theta^* = \sigma_\theta/\sigma_0$, one has here

$$\left.\begin{aligned} \sigma_r^*[\rho, (\rho - 1) +] - \sigma_r^*[\rho, (\rho - 1) -] &= -\frac{1}{\rho}, \\ \sigma_\theta^*[\rho, (\rho - 1) +] - \sigma_\theta^*[\rho, (\rho - 1) -] &= -\frac{\nu}{(1 - \nu)\rho}. \end{aligned}\right\} \tag{50}$$

In contrast, the stress-discontinuities arising from the thermal shock in Danilovskaya's problem [2] of the half-space are independent of the position of the plane wave-front and persist undiminished for all time.

As the inertia parameter γ approaches zero, while ρ and τ are held fixed, the *dynamic* solution to the present problem should tend to the corresponding *quasi-static* solution. Carrying out this cumbersome limit process, which necessitates repeated use of the properties of ψ cited in (17), (18), (19), we obtain:

$$\left.\begin{aligned} u^* &= \frac{1}{\rho^2}\Psi(\rho, \tau), \\ \sigma_r^* = -\frac{2}{\rho^3}\Psi(\rho, \tau), \qquad \sigma_\theta^* &= \frac{1}{\rho^3}\Psi(\rho, \tau) - \varphi(\rho, \tau), \end{aligned}\right\} \tag{51}$$

where

$$\left.\begin{aligned} \Psi(\rho, \tau) &= \int_1^\rho \eta^2\varphi(\eta, \tau)\, d\eta \\ &= \tfrac{1}{2}(\rho^2 - 2\tau - 1)\psi(\xi) - (\rho + 1)(\tau/\pi)^{1/2} \exp(-\xi^2) + \tau + 2(\tau/\pi)^{1/2}. \end{aligned}\right\} \tag{52}$$

Equations (51), (52) are identical with the quasi-static results deduced directly in [10]. As pointed out in [10], the displacement u^* in (51) vanishes identically on the boundary $\rho = 1$, which remains fixed for all time. This curious conclusion fails to apply once inertia effects are taken into account. It should be mentioned that a quasi-static solution (in infinite series form) to the companion problem of a solid sphere whose surface temperature is suddenly altered, was given by Grünberg [15], as early as 1925; the same problem was reconsidered by Melan [16], [17]. Trostel [18] presented a formal quasi-static series solution to the general axisymmetric transient thermal-stress problem for a spherical shell of arbitrary thickness.

Next, consider the limit of (15), (43), (45), (47), as $\tau \to \infty$, for fixed ρ and γ. Here we find, after a tedious computation,

$$\left.\begin{aligned} \varphi = \frac{1}{\rho}, \qquad u^* &= \tfrac{1}{2}(1 - \rho^{-2}), \\ \sigma_r^* = \rho^{-3} - \rho^{-1}, \qquad \sigma_\theta^* &= -\tfrac{1}{2}(\rho^{-3} + \rho^{-1}). \end{aligned}\right\} \tag{53}$$

These values of displacement and stresses also characterize the limit of (51) as $\tau \to 0$. Consequently, the dynamic and the quasi-static solution approach the same steady state. Formulas (53) are in agreement with the appropriate steady-state results for a spherical shell.[11]

Numerical results. Discussion. We turn, finally, to the discussion of certain numerical results based on the dynamic solution to the transient thermoelastic problem treated in this paper. Our main objective, in this connection, is to illustrate the character, and assess the magnitude, of the departures from the analogous quasi-static results. The space and time dependence of the temperature field (15), as well as of the *quasi-static* displacement and stresses (51), were discussed extensively in [10]; the corresponding diagrams need not be reproduced here.

The *dynamic* solution, in contrast to (51), depends not only on ρ and τ, but involves also the inertia parameter γ and Poisson's ratio[12] ν. An inspection of (43), (45), (47), together with the equations defining the various auxiliary functions, reveals that these formulas are expressible exclusively in terms of elementary functions and error functions of *real* argument, provided $\omega = \tau - \gamma(\rho - 1) < 0$. For $\tau > \gamma(\rho - 1)$, however, that is, for points lying behind the wave-front, the displacement and stresses contain, in

[11] See, for example, [19], p. 420. See also [20].

[12] Note that the *dimensionless* quantities u^*, σ_r^*, and σ_θ^* in (51) are independent of ν. See (7).

addition, the complementary error function of *complex* argument $\psi(k^{1/2}\omega^{1/2})$, in which k is the complex parameter (38). According to (16) and (38),

$$\left.\begin{aligned} \psi(k\omega)^{1/2} &= \frac{2}{\pi^{1/2}} \int_x^\infty \exp\,(y^2 - \eta^2 - 2iy\eta)\, d\eta, \\ x = \left[\frac{\omega}{2\gamma}\,(2^{1/2}p^{1/2} - p)\right]^{1/2}&, \qquad y = \left[\frac{\omega}{2\gamma}\,(2^{1/2}p^{1/2} + p)\right]^{1/2}, \end{aligned}\right\} \tag{54}$$

where p and q are given by (28).

While dense tabulations of $\psi(z)$, for real z, are available in [21], the existing tables for complex z appear to be inadequate for our purposes.[13] This fact, along with the general unwieldiness of (43), (45) and (47), complicates the numerical evaluation of the present results. The magnitude of γ is bound to be exceedingly small compared to unity in applications which are of physical interest.[14] It is natural, therefore, to seek

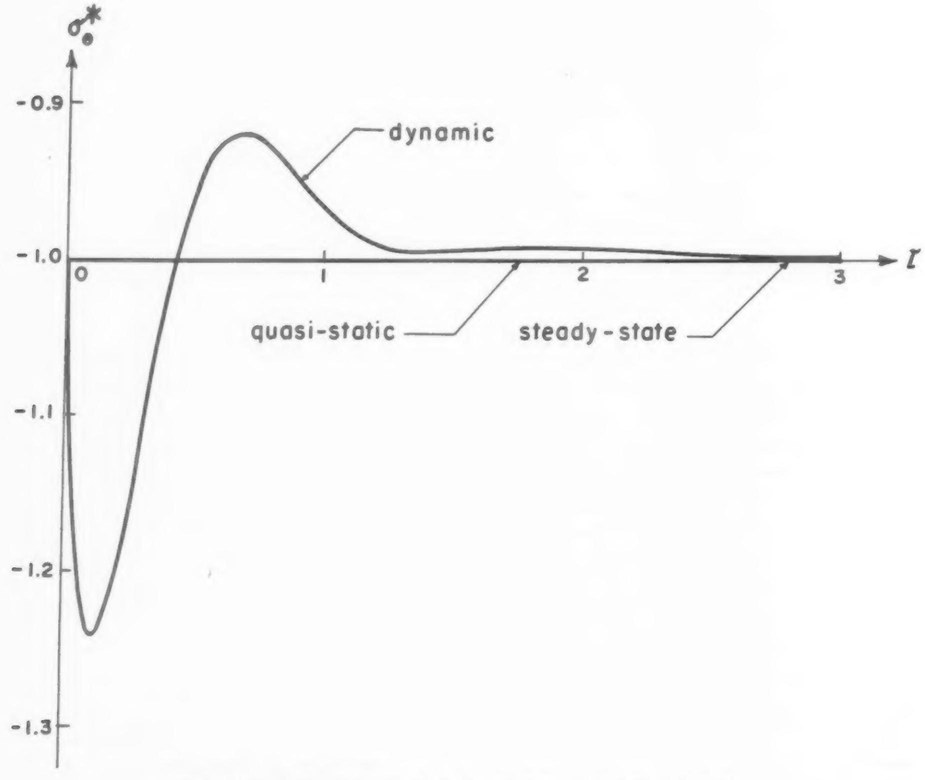

FIG. 1. Time-dependence of σ_θ^* at $\rho = 1$ for $\gamma = 1/5$, $\nu = 1/4$.

[13] Approximate methods for the computation of the error function of complex argument are discussed by Rosser [22].

[14] If $a = 1$ in. and the material is steel, γ is of the order of 10^{-7}.

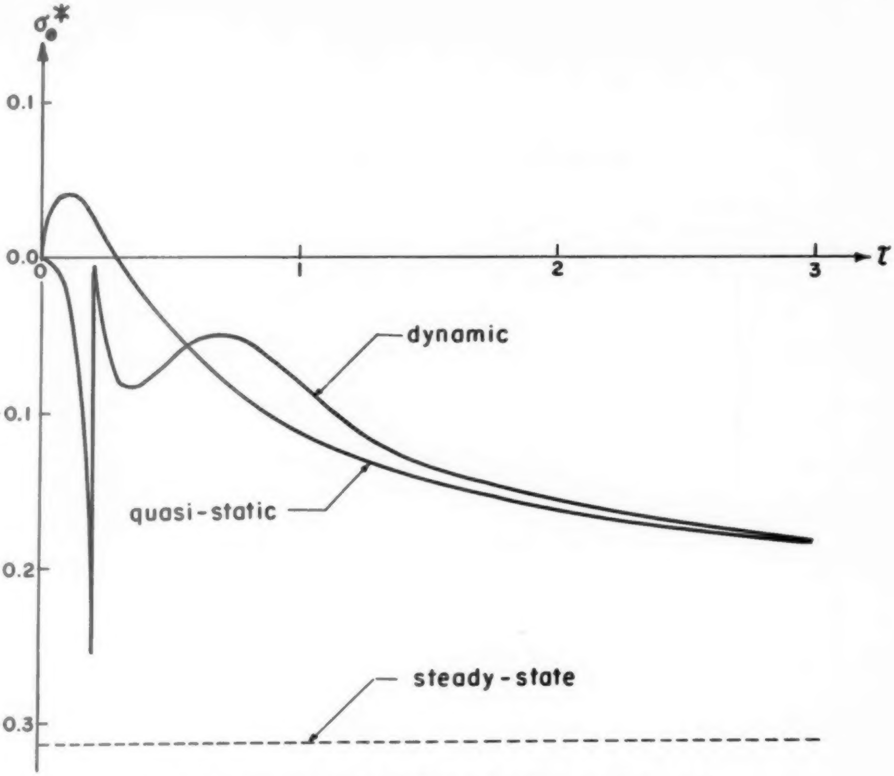

FIG. 2. Time-dependence of σ_θ^* at $\rho = 2$ for $\gamma = 1/5$, $\nu = 1/4$.

asymptotic expansions for u^*, σ_r^*, and σ_θ^*, in the neighborhood of $\gamma = 0$, with the aid of the semi-convergent development (19). Unfortunately, the asymptotic expansions thus reached break down at $\rho = 1$ and at $\tau = 0$; for this reason, they fail to supply useful approximations to the solution in the most significant range of position and time.

Figures 1, 2, 3 show the time-dependence of σ_θ^* at $\rho = 1, 2, 3$, respectively, for $\nu = 1/4$ and $\gamma = 1/5$. This unrealistically large value of the inertia parameter was chosen in order to bring out clearly the *qualitative* nature of the dynamic effects here involved. The underlying computations necessitated appropriate tabulations of the complex error function (54), which were carried out on a 650 I.B.M. electronic computer. The corresponding quasi-static curve ($\gamma = 0$), has been included on each of the diagrams under discussion.

In Fig. 1, the dynamic stress values are seen to undergo a pronounced oscillation, after which they approach the constant quasi-static value $\sigma_\theta^*(1, \tau) = -1$, which coincides with the steady-state boundary-value of σ_θ^*. It is interesting to recall that the boundary-values of the transverse normal stress in Danilovskaya's problem [2], [5], do not exhibit any dynamic effects. In Figs. 2, 3, the dynamic curves display the jump discontinuity (49) at the wave-front and thereafter rapidly approach the corresponding quasi-static curves which, in turn, tend to the common steady-state asymptote. As the shock-wave

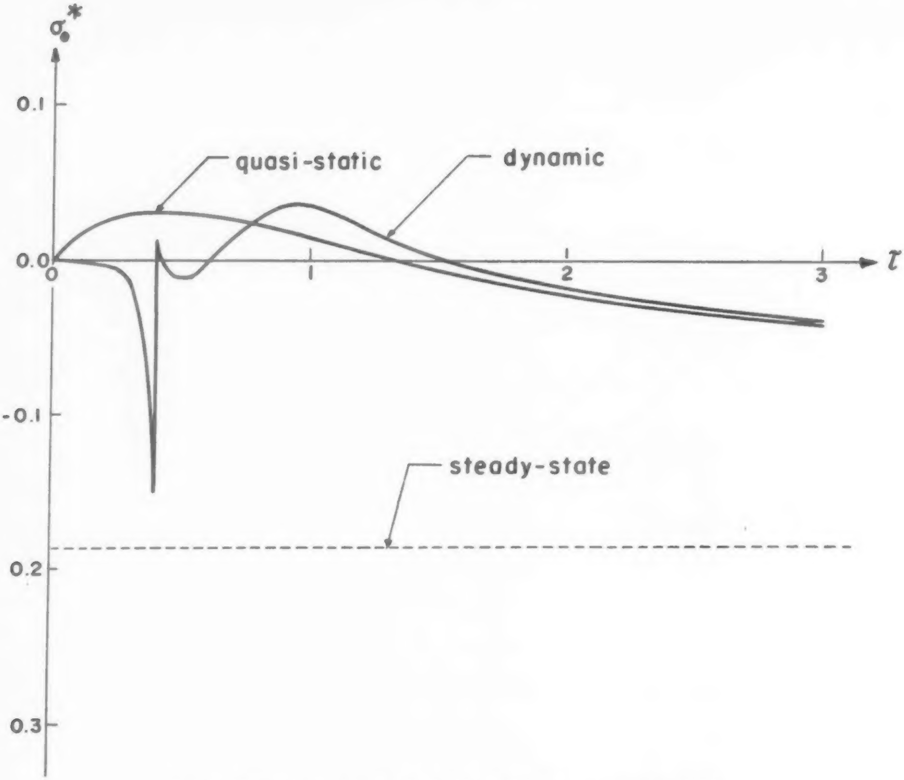

FIG. 3. Time dependence of σ_θ^* at $\rho = 3$ for $\gamma = 1/5$, $\nu = 1/4$.

progresses, the difference between the values of σ_θ^* immediately behind and ahead of the wave-front decreases in accordance with (49); it should be kept in mind that this difference is independent of γ.

Figure 4 illustrates the *quantitative* influence of inertia upon the transient thermal stresses. Here, the dynamic peak-values of σ_θ^* on either side of the wave-front, and the quasi-static values at the wave-front, are plotted as a function of τ. The dynamic curves are based on $\nu = 1/4$ and $\gamma = 7.3 \times 10^{-9}$. This choice of the parameters is descriptive of steel, if $a = 1$ ft. The current value of γ is obtained from (8), (9), by use of the approximate values $\mu = 5 \times 10^{10}$ poundals/ft^2, $\beta = 484$ lbs/ft^3, $\kappa = 1.28 \times 10^{-4}$ ft^2/sec. In this instance $c = 1.75 \times 10^4$ft/sec. Furthermore, for the preceding choice of α and κ, according to (2), $t = 7.8 \times 10^3\tau$.

The curves appropriate to the dynamic σ_θ^* *behind* the wave-front ($\omega = 0+$) and to the quasi-static σ_θ^* *at* the wave-front ($\omega = 0$), are indistinguishable from the coordinate axes for the time-scale underlying the main diagram in Fig. 4. The inset diagram—drawn to a magnified time-scale—exhibits what might be called a boundary-layer effect with respect to time, which is characteristic of the behavior of σ_θ^* during the initial stage of the motion. Thus, the dynamic peak of σ_θ^* ahead of the wave-front is reduced to about one third of its initial limiting value of $-3/2$ during the first 4×10^{-12} second of the

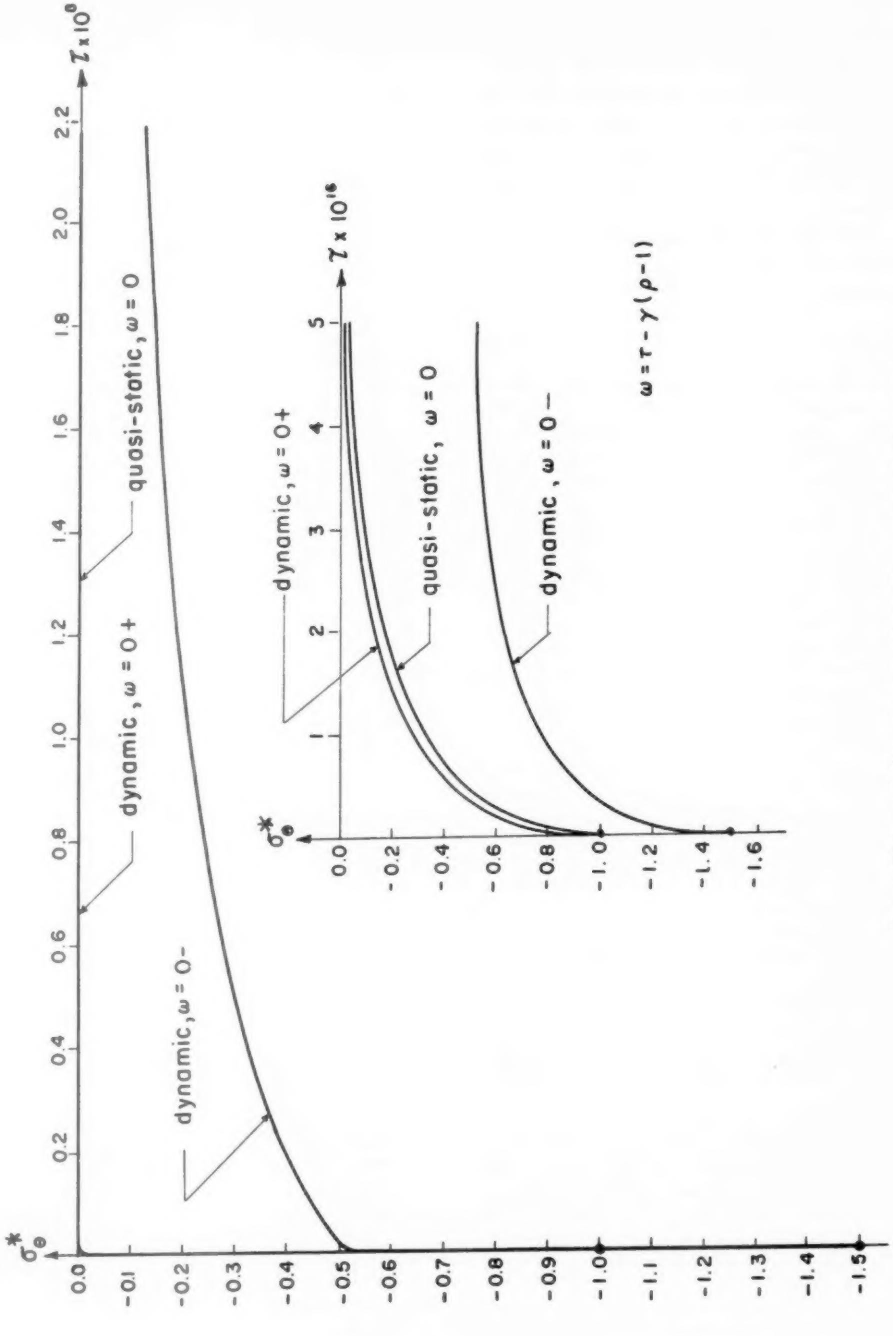

FIG. 4. Values of σ^*_θ at the wave-front for $\gamma = 7.3 \times 10^{-9}$, $\nu = 1/4$.

motion. In the same time-interval, the dynamic peak of σ_θ^* behind the wave-front as well as the quasi-static σ_θ^* at the front of the wave, decrease to less than five per cent of their joint initial value of -1.

In conclusion we emphasize that the present results, which presuppose a step-function dependence upon time of the surface temperature, are bound to require severe modifications once the fiction of an *instantaneous* heating of the boundary is abandoned, even if the rate of heating, though finite, is extremely rapid by physically realistic standards.[15]

Acknowledgment. The authors are indebted to Mr. J. J. Kalker who conducted that portion of the numerical work which was carried out on the I.B.M. 650 electronic computer.

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[15] Compare the conclusions reached in [5].

BOOK REVIEWS

(*Continued from p. 184*)

naturally on the theory of the Hugoniot curve. Polachek and Seeger's article is notable for the balance of its coverage and for the felicitous presentation of experimental records illustrating the theory.

Stever's discussion of condensation phenomena from the point of view of the wind-tunnel man includes both a guide through the kinetic theory of condensation and a description of Oswatitsch and Wegener's theories of shocks and gradual condensation. It is rounded off by a discussion of the experimental evidence and its relation to the theory.

After an illuminating introduction to one-parameter combustion by v. Kármán, Emmons discusses present knowledge on flames. No scope here for a definitive account, so Emmons concentrates instead on inviting the reader to join in the fray! The few topics singled out—flame structure, normal and oblique discontinuous fronts, vorticity production, steady flame geometry and beginnings of a stability theory—are treated with elegance, and the gaps in present knowledge are vividly underlined.

The chapter on detonations by Taylor and Tankin is distinguished from the others by the extensive use of numerical examples to support the physical argument. After a disucssion of the detailed computation of a Hugoniot curve, the viewpoint foreshadowed in Hayes' chapter is taken up that the Hugoniot relation for the detonation front needs to be complemented by a consideration of the unsteady motion of the burned gas behind the front, the possible types of stable front being determined in many cases by boundary conditions far behind the front. The same idea is applied to the case of a shock followed at some distance by a flame front, to explain one of the possible mechanisms of transition from deflagration to detonation, and the discussion of both cases is extended from the plane front to the spherical front moving at a constant speed. The chapter closes with a brief description of the theory of spinning detonation and its experimental support, and of the tentative extension of the gas-dynamical theory to explosions in liquids and solids.

The chapter on rarefied gas dynamics describes the situation at the time of writing and some of the reasons why the situation will be very different very soon. A brief survey of the method and main results of free molecule flow calculations is followed by one of the derivation of the Thirteen Moment and Burnett equations, but Schaaf and Chambré make no secret of their feeling—so well confirmed since—that their study should be postponed in favor of that of the good old Navier-Stokes equations. To give this chapter at least maximum short-term usefulness, they add an extensive survey of present experimental results, and a bibliography which surpasses even those given by the other authors.

With two exceptions, this collection of ten articles meets the specification—that it should not only be, but also remain for many years, the foremost reference work in its field.

R. E. Meyer

Jets, wakes, and cavities. By G. Birkhoff and E. H. Zarantonello. Academic Press, Inc., New York, 1957. xii + 353 pp. $10.00.

This book is a noteworthy essay in bringing together the known theories and facts concerning the flows described in the title, all of which are characterized by the presence of free streamlines or streamlines of discontinuity. For this, if for no other reason, the authors are to be congratulated in filling a real gap in the existing literature of fluid motion.

To quote from the preface "our book draws on the resources of pure and applied mathematics and on experimental physics, and it sheds light on numerous problems of hydraulics and aeronautics. Therefore, it will perhaps have the greatest interest for readers whose scientific curiosity spans all the fields just mentioned. However, we hope that others will also find it a useful and stimulating reference in connection with many special questions".

Owing to the development of high-speed computing methods which enable many calculations to be performed which hitherto would have taken a prohibitive amount of time, approximations are now the order of the day. It may therefore be permissible to say that the present work falls, using the authors' delightful phrase (page 220), "into two roughly equal halves"; the first concerned with exact mathematical methods applied to the two-dimensional motion of an inviscid liquid, the second with the sig-

nificant theory of axisymmetric jets and cavities which has developed in the last two decades, and the reinterpretation of the fundamental facts about vortex trails, and turbulent jets and wakes.

The exact two-dimensional theory of Chapters II-IX proceeds naturally by use of the complex variable and is well-classified to permit the application of uniform methods, but one would like to have seen at least one problem involving gravity flow pursued to a conclusion instead of stopping (page 204) at the point where it becomes interesting. The most valuable contributions of these chapters are contained in Chapter IV where there are brought together accounts of recent work on comparison of flows originated by Lavrentieff and simplified in a series of papers by Gilbarg and Serrin with applications to uniqueness and minimum cavity drag, and in Chapter VII on existence and uniqueness.

The remaining chapters concern (X) axially symmetric flows, (XI) unsteady potential flows, (XII) steady viscous wakes and jets, (XIII) periodic wakes, (XIV) turbulent wakes and jets, (XV) miscellaneous experimental facts. These chapters treat generally increasingly complicated situations. At each stage of complication the authors try to give a picture of what is known, organized as rationally as possible. Here rigorous mathematical theory is not always forthcoming and hypothesis and empirical methods play a large part. Nevertheless, these chapters constitute an admirable and absorbing part of the book.

L. M. Milne-Thomson

Elements of gas dynamics. By H. W. Liepmann and A. Roshko. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1957. xv + 439 pp. $11.00.

Many of us engaged in teaching high speed aerodynamics have eagerly awaited the publication of this book for some time now. It is the opinion of this reviewer that a book as excellent as this one was well worth waiting for. The author's aim was to provide a comprehensive text for senior-first-year graduate study in modern gas dynamics, and in this respect they could not have achieved greater success. The presentation is almost always novel; the material is discussed in the clearest physical and mathematical terms; there is never any loss in rigor; and yet, there is a clarity which makes it ideal for student use. If anything, the student may be led to believe that many of the topics covered are too elementary, however this feeling should be dispelled by the fine set of problems which require much more than just "putting in numbers" to obtain a solution.

Perhaps one of the best features of the book is that it is as modern as the latest research in the field of gas dynamics. Thus, in Chapter 1 the concepts of thermodynamics are introduced including the thermodynamics of reacting and dissociating gas mixtures, a knowledge of which is so essential for the study of hypersonic flow. Chapter 2 considers steady one-dimensional gas flow, and Chapter 3 unsteady one-dimensional wave motion along with a discussion of shock tubes. As throughout, experimental results and photographs are included to give the reader a vivid illustration of the "comparison of theory with experiment." Chapter 4 is primarily concerned with wave and wave interaction phenomena in steady two-dimensional supersonic flow, although more general problems such as the detached shock wave are also considered. Flows in ducts and wind tunnels are considered in Chapter 5, while Chapter 6 outlines methods of measuring supersonic flow phenomena. In addition to discussing the now conventional techniques such as schlieren photography, pitot measurements, etc., the more unconventional and modern X-ray absorption method, direct skin friction technique, and hot-wire methods, are also presented. In Chapter 7 the equations of frictionless flow are derived, along with general flow theorems, and a discussion of the character of the equations of motion. The small perturbation equations and boundary conditions are derived in Chapter 8 for subsonic, transonic, supersonic, and hypersonic flow, and Ackeret's two-dimensional thin airfoil theory is presented. Chapter 9 considers supersonic linearized potential flow over bodies of revolution and a clear distinction, so usually lacking, is made between first-order and slender body theory. All of the similarity laws for the high speed flow regimes are derived in Chapter 10. Chapter 11 is devoted to the elements of transonic flow and, as in the rest of the book, a very clear physical picture is given of the problems involved in this difficult field. In Chapter 12 the method of characteristics is developed by using intrinsic or natural coordinates, so that the discussion is unencumbered by the usual mass of algebra associated with a presentation of the method in cartesian coordinates. An extremely concise but nonetheless clear chapter is 13 which considers the effects of viscosity and conductivity both for laminar and turbulent flow. Here the authors introduce the problems associated with boundary layers, including the effects of dissociation, by the use of the simple and

instructive Couette flow. The most recent work in boundary layer-inviscid flow interaction phenomena are also considered. In the closing Chapter 14 the concepts of gas kinetics of importance to aerodynamics are considered, starting with the fundamental idea of probability concepts, and including such topics as the flow of highly rarefied gases, relaxation phenomenon, and continuum theory limits.

As is always true with a book as comprehensive as this one, there are some minor criticisms. For example, the book would probably have been enhanced by more adequate referencing. In addition, this reviewer would take exception with some of the views expressed in the topics which are now actively being studied, such as boundary layer-induced interactions and continuum theory limits. However, such comments can indeed be considered trivial in the light of the excellence of the book, including, by the way, the format and clarity of printing. Apart from a "must" text in any theoretical aerodynamics course, this reviewer would also recommend the book for anyone involved in high speed aerodynamics research, as well as for those persons with a suitable background who wish to obtain an overall picture of modern gas dynamics. The authors say that they hope this book will be followed by one of a more advanced and specialized nature—we can only look forward to its publication with the greatest interest.

RONALD F. PROBSTEIN

The calculation of atomic structures. By Douglas R. Hartree. John Wiley & ons, Inc., New York, and Chapman & Hall, Ltd., London, 1957. xiii + 181 pp. $5.00.

This book is based on a series of lectures given by the author at Haverford College in 1955. The practical methods of calculating atomic structures are presented in a way to be of immediate help to anyone concerned with such calculations.

Unfortunately, the results of the calculations on atomic structures are not given but a list of references to such results is given in an appendix. Recent work in calculations of this type has been appreciable with thirty or more papers appearing during the past seven or eight years. The fact that no really adequate discussion of these calculation methods has previously appeared in one place should make this book most welcome.

ROHN TRUELL

Rheology, theory and applications. Edited by Frederick R. Eirich. Academic Press, Inc., New York, 1956. xiii + 761 pp. $20.00.

This volume, the first of three intended to provide "integrated surveys" of "well-demarcated areas of rheology", described as "the science of deformation and flow," contains seventeen articles on an impressive range of topics. Most of the work is of an experimental, physical, or chemical nature and need not be described here. Furthermore, most if not all of the articles concerning mechanics or applied mathematics consist in material easily available in other surveys, often by the same authors, and it is possible that the main virtue of the volume is to put between two covers a number of different conceptions of this fluid science.

"Phenomenological macrorheology" by M. Reiner and "Dynamics of viscoelastic behavior" by Turner Alfrey, Jr., and E. F. Gurnee deal with what seems to be the same topic from the mechanical point of view and differ in that the former emphasizes terminology, the latter, the consequences of linearity and the associated operator techniques. While both articles contain passages filled with tensor equations, the methods and problems are essentially one-dimensional, and prominence is given to the devising of complicated materials by imagined couplings of simpler materials.

Only three articles approach the mathematical theories of genuine large deformation and flow from a standpoint that seems tenable in view of recent researches in general mechanics. J. G. Oldroyd's "Non-newtonian flow of liquids and solids" deals mainly with rectilinear flows. As is now well known, specialization of correct three-dimensional equations to such flows does not yield the same results as were assumed in the purely one-dimensional treatments of the older studies on rheology. The article of Oldroyd, however, concentrates upon its author's researches of some years ago and cannot be regarded as giving a just view of the field today. While the general equations for an incompressible fluid are the author's equations (46), viz.

$$p_{ik} = \eta D_{ik} + \tfrac{1}{2}\psi \sum D_{ij} D_{jk} - p\delta_{ik} ,$$

virtually all of the article is restricted to the case $\psi = 0$; that is, to a fluid which might be described as devoid of cross-viscosity though possessed of variable shear viscosity. Rivlin's general solutions for $\psi \neq 0$ are alluded to but not described. Moreover, more recent researches of Ericksen[1] and Stone[2] indicate that rectilinear flow generally fails to exist if $\psi/\eta \neq$ const.; in a special case, Green and Rivlin[3] have obtained an approximate solution in which a circulating cross flow is superimposed upon the flow down the tube.

William Prager's "Finite plastic deformation" is an interesting exception to the usual literature of plasticity in that it attempts to select those theoretical problems which may be interpreted as involving large deformation. Except for a few pages at the end, only the case of a body rigid up to yield and perfectly plastic thereafter is considered; even here, "finite" must be qualified, since the inertial terms in the equations of motion are neglected. This article is very clearly written and is illuminating for its well chosen discussion, examples, and figures. Most of it concerns plane flows, and the complicated calculations are omitted in favor of explanation of the results in a form comprehensible to those who are not specialists in plasticity.

"Large elastic deformations" by R. S. Rivlin attempts a task less ambitious than those set themselves by the other authors, since it is devoted to a definite and well poised problem of mechanics. Its value lies principally in explaining to those who might wish to apply it the general theory of elasticity in the form and status a considerable body of specialists now see for it. The article presents only the parts of the theory where the author has made notable contributions; this results in a certain lack of appreciation of other aspects, and in particular it seems to me that the work of Signorini does not deserve to be dismissed along with that of Seth on p. 353, and that the general approximation process initiated on pp. 382–383 was given, substantially, twenty years ago by Signorini. The reader of Rivlin's article will come away with an accurate summary of certain definite predictions of finite elasticity, but if he has no more mathematical apparatus than the author requires of him, it is unlikely that he can read further in the literature of the subject.

Another exceptional article, of a very different kind, is "The statistical mechanical theory of irreversible processes in solutions of macromolecules," by J. Riseman and J. G. Kirkwood. This is a condensed extract from one aspect of the group of statistical theories proposed by Kirkwood and his students. A much higher level of mathematical and physical preparation is needed for the reader of this article, which concerns statistical mechanics of chains and employs a Riemannian geometry for the subspace obtained by imposing constraints on the motions in the molecular configuration space. Approximation procedures are introduced to get definite results for special models.

Looking back at the volume as a whole, while we find much overlap of qualitative discussion, especially regarding names and symbols, at the same time this heavy tome seems to have found no space to present the recent exact studies of visco-elastic and time rate processes. The fundamental memoir of Oldroyd[4] is cited thrice (pp. 51, 657, 676), but in such a way that one not already familiar with it could scarcely divine its subject; the memoirs of Noll[5] and of Rivlin and Ericksen[6] perhaps appeared too late for mention. However, it seems unlikely that fundamental problems of large deformation can ever be described successfully, even *a posteriori*, at the mathematical level most of the surveys in this volume strive to maintain.

[1]Q. Appl. Math. **14**, 318–321 (1956).
[2]Q. Appl. Math. **15**, 257–262 (1957).
[3]Q. Appl. Math. **14**, 299–308 (1956).
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[5]J. Rational Mech. Anal. **4**, 3–81 (1955).
[6]J. Rational Mech. Anal. **4**, 323–425 (1955).

C. Truesdell

Experimental designs in industry. Edited by Victor Chew. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1958. xi + 268 pp. \$6.00.

During recent years experimental designs have been employed at an increasing rate in industrial research and development work. The various classical designs as Latin squares, balanced incomplete block designs, etc., although they have their place in industrial research are quite inadequate for many problems that arise in industry where often the number of factors and the number of levels of each

factor are far too large to work with complete replications. The fractional replications introduced by Finney in 1945 proved particularly useful in this situation. The methods of fitting response surfaces first introduced by Box and Wilson in 1951 seem to have proven their value in situations where the factors may be applied in continuously varying levels.

It was thought that these designs had been used for a long enough time to exchange information on experiences with their use. To this purpose a symposium was held at North Carolina State College on November 5–9, 1956. The present volume contains a selection of papers presented at this symposium. The first paper by V. Chew gives a general survey of the basic ideas of the analysis of variance. The next paper by R. L. Anderson deals with complete and fractional factorials and the technique of confounding. This is followed by a paper on multiple regression analysis by R. J. Hader and A. H. E. Grandage. An exposition by J. E. P. Box and J. S. Hunter of the methods of fitting response surfaces introduced by Box concludes the theoretical part of the volume. The second part reports on experiences in the application of designs in industry in papers by W. S. Connor, W. H. Horton, M. B. Carroll and O. Dykstra Jr., De Baun and A. M. Schneider and concludes with a report by C. A. Bicking on experiences with designs and needs for designs in ordnance experimentation.

Although all papers presented will be of interest to anyone in the field of mathematical statistics, the exposition of the methods of fitting response surfaces seems of particular significance. It seems to the reviewer that these methods introduce ideas that are not only basically new but have also arisen from a very urgent practical need.

It is of course not the purpose of the papers presented in this volume to give a complete account of the mathematical foundations of the methods presented. The authors therefore restrict themselves to a presentation of the calculation procedures and a description of the situations in which the methods apply. Numerous applications to problems that have actually arisen in industrial research are presented. For those who are interested in a more thorough study a large index of references is given at the end of each paper.

H. B. Mann

Principles of noise. By J. J. Freeman. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1958. x + 299 pp. $9.25.

This book is designed to introduce graduate students, principally in electrical engineering, to the fundamentals of noise analysis. Thus, the mathematical tools—Fourier series and integrals, probability theory, stochastic processes—are developed in sufficient detail for the purpose and the physical sources of noise are described before proceeding to their analysis. The work may also serve as an elementary introduction to several recent more advanced treatises on random signals and noise. There are many well chosen examples and problems for the reader.

W. F. Freiberger

The preparation of programs for an electronic digital computer. By Maurice V. Wilkes, David J. Wheeler, and Stanley Gill. Addison-Wesley Publishing Co., Reading, Mass., 1957. xiv + 238 pp. $7.50.

When the first edition of this work was published in 1951 it represented the first effort to make the art of programming accessible in bookform and formed an invaluable aid in spreading knowledge of a relatively new field. Since then, developments have been rapid and books on computers are rolling off the presses. Thus, this second edition of a pioneering treatise inevitably seems somewhat outdated. The language in which the programs are presented is still the order code of the EDSAC and although material on other machines is included, it is this code the reader will have to master to understand the book. One of the most striking developments of recent years, at least in America, has been the increasing use of automatic coding systems, and there is no discussion of this interesting and important subject in the book.

This is, thus, strictly a book for the specialist who will make the effort of mastering it for the sake of the innumerable hints on methods and techniques strewn throughout its pages; the authors' wide experience vouchsafes their effectiveness.

W. F. Freiberger

TENTH INTERNATIONAL CONGRESS OF APPLIED MECHANICS

The *Tenth International Congress of Applied Mechanics* will be held in the Congress Building at Stresa (Italy) from *Wednesday, 31 August through Wednesday, 7 September 1960.*

Apart from a number of invited general lectures the technical sessions of the Congress will be held in two sections, viz.:

Section 1: Fluid dynamics (hydrodynamics and aerodynamics).
Section 2: Mechanics of solids (rigid body dynamics, vibrations, elasticity, plasticity and theory of structures).

It should be noted that thermodynamics and computational methods as such are not included, although specific applications of computational methods to pertinent problems of one of the two sections mentioned above are acceptable subjects for papers to be read at the Tenth Congress.

Previous Congresses have demonstrated the desirability of an adequate period of time for the presentation and discussion of individual papers. In order to allow a period of 45 minutes for each paper (30 mins. for presentation and 15 mins. for discussion) a Program Committee will make a selection from papers submitted for presentation. Abstracts of papers should be submitted *in four copies to the Secretary of the International Committee* (Prof. Mekelweg 2, Delft, Netherlands) *before 1 January 1960.* Preferably they should not exceed two typewritten pages (double-spaced) and in no case should they exceed four pages. In order to facilitate the work of the Program Committee it is recommended that abstracts be in *two* of the official Congress languages (English, French, German and Italian). Authors are urged to make their abstracts as clear as possible, since selection of papers has to be based upon them. *Decisions of the Program Committee are final,* and it will be understood that it is impossible to enter into correspondence about them with authors of papers. They will be informed promptly of the decision on each paper.

Day-to-day organization of the Congress is effected by the Italian Organizing Committee (President: Prof. G. Colonnetti; Secretary: Dr. F. Rolla, *Consiglio Nazionale delle Ricerche, Ufficio relazioni internazionali, Piazza delle Scienza 7, Roma*). *All correspondence* (apart from submission of papers) *should be addressed to* the *Italian Organizing Committee.* Information on accommodation, also registration forms, will be obtainable from Dr. Rolla on and after 1 September 1959.

The Executive Committee of the
International Committee for the
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The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. [illegible]

[illegible]

MATHEMATICAL METHODS OF OPERATIONS RESEARCH

By Thomas Saaty, Mathematician in the Executive Office of the Secretary of the Navy. 432 pages, $10.00

A graduate text in the field of [illegible] and [illegible] [illegible] analysis are presented [illegible] application [illegible] the [illegible]

LINEAR PROGRAMMING: Methods and Applications

By Saul I. Gass, U. S. Department of Agriculture Graduate School, Washington, D. C. 223 pages, [illegible]

The first text on the subject, it offers a basic presentation of the theoretical, computational, and applied areas of linear programming, emphasizing the problems it can solve and the importance of techniques to formulate and solve such problems. The revised simplex method, parametric linear programming, and a summary of digital computer codes are covered.

A PRIMER OF PROGRAMMING FOR DIGITAL COMPUTERS

By Marshal H. Wrubel, Indiana University. McGraw-Hill Series in [illegible]

An introductory text designed for undergraduate courses for physical scientists, engineers, and all other students [illegible] "The book is clear and [illegible] [illegible]"

COMPUTABILITY AND UNSOLVABILITY

By Martin Davis, Rensselaer Polytechnic Institute. McGraw-Hill Series in Information Processing and Computers. [illegible] pages, [illegible]

Concerned with the [illegible] theory and the applications of the subject to algebra, number theory, and symbolic logic. [illegible] of the [illegible] contributions in the computer field.